Revenue Sharing in Airline Alliances

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Abstract

Airline alliances as means of collaboration among independent carriers are a growing trend in the industry. From a revenue management perspective, one of the most significant features of the alliances are codeshare itineraries by which independent airlines can collaboratively market and operate flights. Different from traditional, monopolistic airline revenue management, alliance members control a decentralized network of resources through independent reservation and information systems.

To study the revenue management problem of such decentralized network environment, we propose a two-stage hierarchical approach. In the first stage, airlines agree on how to share the revenues generated by these interline products. In the second stage, airlines operate independent inventory control systems in order to maximize their own expected revenues. Through both analytical and numerical studies, we find that the choice of the revenue sharing rule has a great impact on the performance of the alliance. In particular, in the static setting where each airline uses partitioned booking limits, there exists a revenue sharing rule under which the decentralized system can do as well as the centralized system. We further construct an asymptotic regime in which airlines’ capacities and demands grow proportionally large, and prove that under our proposed revenue sharing rule, the performance of the alliance under dynamic inventory control converges to the performance under static booking limit control. The numerical comparisons between several dynamic heuristic policies and the static booking limit control confirm the quality of the approximation.

Nevertheless, given that the revenue sharing rule that is provably optimal in our model requires to disclose private demand information, we propose a simple revenue sharing rule that is based on public fares. This simple heuristic performs noticeably well in our numerical experiments, becoming an interesting candidate to be pursued in practice.

Key words: revenue management, capacity control, contract design, noncooperative game theory, Nash equilibrium.

1. Introduction

An airline alliance is an agreement between multiple independent partners to collaborate in various activities to streamline costs (e.g., by sharing sales offices, maintenance facilities, ground handling
personnel, check-in and boarding staff, etc) while expanding global reach and market penetration. The presence of alliances in the airline industry has followed an increasing trend since the first large airline alliance was formed in 1989 between Northwest and KLM. By March 2009, the three major alliances (Star, Sky Team and Oneworld) combined flew around 73% of all passengers worldwide. The aggregate number of members evolved from 33 in 2003 to 52 in 2010. Limited by restrictions on mergers with foreign partners due to sovereignty and regulatory issues, but at the same time losing billions of dollars amid volatile fuel prices and a pullback in spending, U.S. airlines are seeking to expand their alliances and trying to extend synergies within current partnerships (e.g., see Chakravorty (2010)).

One of the fundamental building blocks of an airline alliance is the ability to code-share flights. Code sharing or codeshare is the term used to describe the practice of multiple airlines selling space on the same flights, where flight segments that are operated by one carrier can be marketed for sale by other one(s). Even though this practice is older than the formation of the major alliances and may even occur between airlines outside of an alliance, its extent is an order of magnitude higher between airlines inside an alliance (Vinod (2005)). For example, within Oneworld, a one-stop flight from New York (JFK) to Barcelona (BCN) can be operated by two airlines with the first leg New York (JFK) to Madrid (MAD) operated by American Airlines and the second leg Madrid (MAD) to Barcelona (BCN) operated by Iberia. This (interline) itinerary, under codeshare agreement, is marketed by both American Airlines and Iberia, and generates revenues for both carriers, which will depend on the price paid by the passenger to the marketing airline for the ticket, and a transfer price paid by the marketing airline to the operating airline for the consumption of capacity. As the airline Revenue Management (RM) practice within the boundaries of a company is acknowledged to generate incremental revenues when switching the inventory controls from leg-based to origin-destination (O-D)-based, the formation of alliances scales up this phenomenon. For instance, Oneworld acknowledges more than $3 billion dollars as incremental revenues derived from alliance synergies and $2.8 billion for interline revenues in 2008.

Certainly though, with the advent of airline alliances, the practice of O-D RM is even more challenging. Within alliances, each airline member wants to maximize its own revenues by ensuring the optimal traffic flow in the global network, whose segments are now operated by several, independent carriers. Thus, an itinerary, as a combination of flight legs, may be operated by multiple companies, so that a request for a ticket will be accepted only if all airlines operating the legs agree to do so. In the current RM practice, reservation systems of alliance members are still quite independent due to organizational and technical discrepancies, alliance exit-options, and revenue sharing and antitrust immunity considerations. Therefore, airline alliances among members representing their own interests induce the design of incentive and coordination mechanisms. The most prevalent mechanism in the airline industry is to use soft blocks (see Vinod (2005)), where the operating carrier dynamically accepts/denies requests from the marketing carrier. Another possible way to manage interline itineraries is through blocked seat allotments. In this scenario, the marketing carrier purchases a block of seats at a pre-determined price from the operating carrier. At a predetermined time before departure, the marketing carrier will release the unused hard block space to the operating carrier.

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1 According to Vinod (2005), these incremental revenues may be in the range 2%-7%.
2 Another possible way to manage interline itineraries is through blocked seat allotments. In this scenario, the marketing carrier purchases a block of seats at a pre-determined price from the operating carrier. At a predetermined time before departure, the marketing carrier will release the unused hard block space to the operating carrier.
agement of soft blocks: We study contracts among airline partners that define the transfer prices between the marketing and the operating carriers.

Revenue sharing contracts have a two-side effect on airline alliances members. In principle, each airline might want to get as much share from the interline flights as possible, because it directly reflects the gain from codeshare. However, the revenue generated from interline flights is not only determined by the proportion of the codesharing fare that an airline can get, but also by the volume of codeshare ticket sales. For instance, under a very asymmetric fare split, the weak partner could often be the bottleneck for accepting a particular interline itinerary. Therefore, it is very important to have a well rounded view of the impact that these revenue sharing agreements have on the operations of the alliance when defining transfer prices. A smart mechanism should provide large enough shares (marginal revenues per ticket) and at the same time create incentives for the partners to collaborate on interline itineraries.

The rules used in practice to split revenues include fixed transfer prices for particular itinerary-fare-class combinations, or other splitting rules related to fixed proration rates based on relative local fares published by the interline partners, or on relative mileage. Transfer prices are governed by special proration agreements (SPAs) that are privately negotiated by airline partners. Within the International Air Transport Association (IATA), there is a Prorate Agency that maintains the passenger and cargo prorate agreements on behalf of airlines signatory to the Prorate Agency Agreement. Part of the functions of the Prorate Agency since September 2008 is to publish a Prorate Manual Passenger (PMP) quarterly, which includes agreed weighted mileage formulas. From an operational point of view, these splitting rules are static in the sense that they do not change as capacity of the carriers is depleted and time passes, and are therefore of dubious revenue performance. According to Wright et al. (2010), major airlines are trying to circumvent this by considering dynamic schemes, such as exchanging bid-prices, but there are technical and legal barriers to implementation like antitrust legislation in U.S. and lack of trust among partners. In summary, it is not clear nowadays which practice will prevail in the future.

Although we are not the first to investigate the operations of an airline alliance (we review the literature in the following section), to the authors’ best knowledge this is the first paper that theoretically analyzes and characterizes equilibrium strategies for airline alliance members under various revenue sharing parameters in a general network topology. Specifically, we propose a two-stage hierarchical approach. In the first stage, airlines negotiate and agree on a revenue sharing contract. In the second stage, given the revenue sharing contract, each airline manages its independent reservation system to maximize its own revenue.

In our stylized model, each airline manages static partitioned booking limits (SBL), which splits the capacity among the different itinerary-fare-class combinations (products). But different from the traditional, monopolistic RM models, the airline is allowed to share capacity with alliance partners in order to maximize expected revenues. So, essentially, alliance members are collaborating and

3However, note that straight application of mileage-based proration may lead to a priori unfair revenue allocations that may deter a carrier from accepting interline passengers. For instance, under a mileage-based proration scheme a $782 one-way ticket between JFK and Barcelona would be split in $719 for American and just $63 for Iberia.
competing at the same time. We follow a two-stage hierarchical approach to model this setting. In the first stage, airlines agree on how to share the revenues generated by the codesharing products by defining fixed proration rates. We do not study the negotiation process here, but rather analyze the impact of the outcome of that preliminary process. In the second stage, airlines operate independent inventory control systems in order to maximize their own expected revenues. To capture the interaction among players in the second stage in this decentralized network, we focus on strategies that are Nash equilibria. Technically, to solve the second-stage noncooperative game, we decompose the big fixed-point problem with multiple players into a collection of independent, simpler sub-problems, and characterize the set of Nash equilibria under fixed proration rates. Our findings show that the game played by the airlines exhibits strategic complementarity in that higher level of capacity sharing among the solutions in the Nash equilibria set results in higher revenue for each alliance member, and no collaboration leads to the worst possible outcome. Nevertheless, the Nash equilibrium that will be achieved depends on each airline’s beliefs about other airlines’ willingness to collaborate.

Next, proceeding backwards to stage one in our analytical approach, we derive closed-form, fixed proration rates that can achieve the same performance of the centralized system, i.e., of a system run by a central planner that controls the whole global network. We provide intuition for why revenues should be shared in such a way among alliance members so that the system can perform optimally.

Our work may have both theoretical and practical implications. First, a static revenue sharing mechanism is a good reflection of the allocation of revenue generated by codesharing flights and can provide intuitions for a more complicated dynamic mechanism. Second, using the right static revenue sharing mechanism, and applying equilibrium static booking limit controls but in a real (dynamic) setting, can perform noticeably well. This is supported theoretically by an asymptotic optimality result, and is numerically verified in our experiments. Moreover, we show that the relative gain that an airline can get by unilaterally deviating from using our proposed static heuristic policy is marginal. Third, static transfer price schemes are arguably the easiest to implement and, as mentioned above, they have been widely used in the industry practice. In this respect, our results can be viewed as providing a theoretical support for the use of these simple revenue sharing rules. Finally, it is worth pointing out that our model and results can be extended beyond the scope of airline alliances to systems satisfying the following conditions: (a) multiple agents own sets of resources with finite capacity, (b) products are defined as a subset of resources, and agents must decide how much of each resource capacity in their control to allocate to these products, and (c) the production (supply) of a given product is determined by the minimum capacity that is allocated to the set of resources required in its production.

1.1 Related literature

Quantity-based, network RM involves controlling a fixed and perishable capacity of a network of resources over a finite horizon, with the objective of maximizing revenues. Products (i.e., itinerary-fare-class combinations) are defined over the network, spanning one or more resources (i.e., legs). There is an extensive literature on monopolistic RM, where a single airline owns all the resources, and a variety of static and dynamic (i.e., depending on remaining time and capacity) control methods
have been proposed. We refer the reader to the books by Talluri and van Ryzin (2004) and Phillips (2005) for an in-depth coverage of this work.

In contrast, the existing literature on airline alliance collaboration and operation is quite limited. A major challenge faced by academic researchers and industry practitioners is the conflict between alliance initiatives to collaboratively manage code share flights and the independent control of individual airline reservation systems, as pointed out in the overview papers by Boyd (1998) and Vinod (2005). Since a perfect integration of revenue management and reservation systems of alliance members is impractical and may be subject to regulatory constraints, airline alliances, at the operational level, actually behave like decentralized systems. However, little attention has been paid to the game theoretical nature of airline alliances.

Netessine and Shumsky (2004) study a static model for seat allocation under both horizontal and vertical competition between two airlines for one- and two-leg networks. The horizontal competition occurs on the single leg case, while vertical competition refers to the tension between local and connecting passengers in a two-leg network. On the strategic level, they compare the allocation under a decentralized control to the benchmark obtained by a centralized control. They show that codesharing can lead to a suboptimal seat allocation if revenue from the connecting itinerary is split proportionally according to the local fares. Their model is less general than ours though, not just because of the specific topology, but also because they assume complete information, do not allow overlapping markets for vertical competition (i.e., only one airline could serve a given O-D pair), and the revenue from connecting passengers is defined as the sum of the local fares, which is not usually the case in practice.

Wright et al. (2010) study a two-airline alliance network and formulate the problem as a dynamic game: a transfer price is computed at each customer arrival epoch, and each airline dynamically controls its inventory to maximize its own revenue during the booking process. They provide a counter example to show the non-optimality of Markovian transfer price schemes, derive equilibrium acceptance policies using the value functions from each airline’s dynamic program, and conclude that the performance of each transfer price scheme and partner’s behavior is significantly affected by the proportion of local requests in the total demand. The work of Wright et al. (2010) highlights the complexity of computing dynamic revenue-sharing policies in a decentralized environment, and most of their results are derived numerically. In contrast, using static booking limits (SBL), we are able to characterize optimal strategies for an arbitrary alliance in terms of its network topology and number of airlines. In a different context, Agarwal et al. (2009) study the fairness issue of revenue allocation among cargo carriers. They propose a cooperative game approach to measure various benefit/cost allocations for alliances in sea cargo and air cargo industries. Within the RM context, the distinguishing features of our proposal are: (a) general network topology and demand distributions, (b) a two-stage hierarchical framework that allows to decouple the contractual side (i.e., establishing the proration rates) from the operational side (i.e., inventory control), (c) characterization of optimal proration rates that can be implemented in a decentralized setting under incomplete information.

Other than the RM side, various issues of airline alliances have been examined by researchers of different fields. Oum and Park (1997) describe the government policy towards airline alliances. Park
(1997) examines the effects of airline alliances on profits, air fare, and economic welfare. Park and Zhang (2000) and Bamberger et al. (2004) empirically investigate the effects on air fares, passenger volume, and consumer surplus of international and domestic airline alliances, respectively. More recently, Gayle (2007) analyzes how policy makers can use a structural econometric framework to quantify the competitive effects of proposed code-share alliances, in which potential alliance partners compete on overlapping routes in the pre-alliance industry. He applies the framework over the consumers’ convenience of the Delta/Continental/Northwest alliance, and does not find any significant change between post and pre-alliance fares. Czerny (2009) questions the overall usefulness of code-share agreements from a welfare perspective, in the sense that interline passengers always benefit from these agreements while non-interline passengers are worse off. Wan et al. (2009) suggest that the changes in fare values on parallel routes may be either insignificantly different from the pre-alliance setting, or may even drop. Brueckner and Proost (2010) analyze carve-outs under airline antitrust immunity, which are designed to limit the potential anticompetitive effects of cooperation by alliance partners in hub-to-hub markets, where they provide overlapping nonstop service.

Our paper also touches on the topics of decentralized pricing and revenue sharing. The idea that the price system has a significant impact on the allocation of resources dates back to the work on “the theory of optimum allocation of resources” by Kantorovich and Koopmans. Kantorovich (1965) shows how the possibility of decentralizing decisions in a planned economy depends on the existence of a rational price system. In their general equilibrium theory, the solution (a utility allocation to agents) to any maximizing linear social welfare function is a Pareto optimal allocation. Moreover, according to the second fundamental theorem of welfare economics, there exists a price vector such that the allocation is an equilibrium in which each agent maximizes his utility, and market clears. Our results can be viewed as an application of these ideas to the seat allocation problem in alliance network revenue management. Further discussion about general equilibrium could also be found in Mas-Colell et al. (1995).

Revenue sharing has been studied in the Operations Management field as a contract type to achieve supply chain coordination. Motivated by the practice in the videocassette rental industry, Cachon and Lariviere (2005) study revenue sharing contracts between a supplier and a single fixed-price or price setting newsvendor, and demonstrate that this type of contract coordinates the supply chain (i.e., it achieves the profits of a central planner). They also prove that revenue sharing coordinates a supply chain with symmetric retailers competing in quantities, or competing newsvendors with fixed prices. Our setting is different from theirs in that we consider asymmetric players within a network environment offering multiple products that consume multiple resources.

1.2 Organization

The rest of our paper is organized as follows: In Section 2, we describe our static network model, and airlines’ strategies to control inventories in terms of SBL. Section 3 exposes the analysis of the two-stage hierarchical approach, starting backwards from the noncooperative game related to the

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4We refer the reader to Cachon (2003) for a discussion of several contract designs in the context of (manufacturing) supply chain management.
inventory control problem that takes place in the second stage, and following with the selection of proration rates in the first stage. Since the optimal proration rates requires to disclose private information about demand for local itineraries, we also present here a heuristic proration rule that is just based on publicly available fares. The asymptotic optimality of using together the optimal fixed proration rates and SBL to control capacity is discussed in Section 4. Our results are tested numerically in Section 5, and our concluding remarks are presented in Section 6. Proofs can be found in Section A3 in the Appendix.

2. Model

We consider a general alliance network with \( K \) airline companies, \( m \) legs (or resources) and \( n \) origin-destination (O-D) itineraries (i.e., products). We denote by \( K \) the set of airlines and we index by \( k = 1, \ldots, K \) an arbitrary member of \( K \). Similarly, we denote by \( M \) and \( N \) the sets of legs and itineraries, which we index by \( i = 1, \ldots, m \) and by \( j = 1, \ldots, n \), respectively. Each itinerary (or product) is characterized by its route (i.e., the set of legs that it uses) and its fare. We assume that each resource is operated by a single airline and that each itinerary consumes one unit of capacity (one seat) on each leg of its route.

We let \( M_k \) be the subset of legs operated by airline \( k \) so that \( \{ M_1, \ldots, M_K \} \) forms a partition of \( M \). We also define \( N_k \) to be the set of itineraries that use at least one leg in \( M_k \). The set \( N_k \) is partitioned into two subsets \( N_L^k \) and \( N_I^k \). The itineraries in \( N_L^k \) are those that belong to the Local network of airline \( k \), i.e., they are operated and marketed exclusively by airline \( k \). On the other hand, \( N_I^k \) is the set of itineraries in the Interline network in which airline \( k \) participates either as operating or marketing carrier; that is, those itineraries that use at least one leg from another airline. In general, the superscripts ‘L’ and ‘I’ will be used to denote quantities associated to the Local and Interline networks, respectively.

Some additional notation and conventions that we will use to model the structure of the alliance follow:

- \( K_j \): subset of airlines that operate at least one leg in itinerary \( j \).
- \( L_j \): subset of legs used by itinerary \( j \).
- \( A^L \in \{0,1\}^{m \times n} \): incidence matrix for the network of local itineraries: \( A^L(i,j) = 1 \) if and only if itinerary \( j \in N_L := N_L^1 \cup \cdots \cup N_L^K \) uses leg \( i \in M \). We also define the sub-matrix \( A^L_k \) associated to the itineraries and legs operated exclusively by airline \( k \): \( A^L_k(i,j) = 1 \) if and only if itinerary \( j \in N_L^k \) uses leg \( i \in M_k \).
- \( A^I \in \{0,1\}^{m \times n} \): incidence matrix for the network of interline/codeshare itineraries: \( A^I(i,j) = 1 \) if and only if itinerary \( j \in N_I := N_I^1 \cup \cdots \cup N_I^K \) uses leg \( i \in M \). The filtered matrix \( A^I_k \) is defined analogously to \( A^L_k \).
- $D_j$: cumulative demand for itinerary $j$ during the planning horizon. We assume that demand for different itineraries are independent random variables with finite mean $\mu_j$ and continuously increasing c.d.f. $F_j$, for $j = 1, \ldots, n$. We also define $\bar{F}_j := 1 - F_j$ to be the right tail distribution of $D_j$.

- $p_j$: fare of product $j$.

- $\beta_{kj}$: fraction of the fare $p_j$ of itinerary $j$ collected by airline $k$. We denote by $\mathcal{B}$ the set of feasible proration rates; that is, $\mathcal{B} := \{\beta \geq 0$ such that $\sum_{k \in \mathcal{K}} \beta_{kj} = 1$, for all $j \in \mathcal{N}\}$.

- $C$: vector of leg capacities, partitioned in $C_1, \ldots, C_K$ sub-vectors of capacities from legs operated by corresponding alliance members.

- Conventions: Matrices and vectors are denoted in bold letters and their components are denoted using the corresponding unbold letter. For example, $X$ is a matrix with components $X(i, j)$ or $X_{ij}$. Whether a variable is a matrix or a vector will be clear from the context. For a given vector $Y$, the sub-vector $Y_k$ refers to the components of $Y$ associated to airline $k$. For example, $C_k$ is the vector of capacities of the legs operated by airline $k$ and $C_k(i)$ is the number of seats available in leg $i \in \mathcal{M}_k$. In order to maintain notation reasonably simple, we keep the original labeling of the resources and products within a sub-matrix.\(^5\) The cardinality of a set $S$ is denote by $|S|$ and the transpose of a matrix is denoted by a prime (‘). The indicator function of an event $E$ is denoted by $\mathbb{I}(E)$.

It is worth noticing that the structure of the alliance is flexible enough to allow for: (a) Multiple fare classes on the same origin-destination route, which are identified as different products, (b) Multiple airlines in direct competition on the same routes, since two different legs can serve the same O-D pair, and (c) Airlines collecting revenues on itineraries operated exclusively by other airlines. For instance, going back to our example in Section 1, Iberia could sell a single-leg ticket in the route JFK-MAD operated by American (i.e., a codeshare itinerary). We will, however, slightly narrow the generality of the model by imposing some minor restrictions which we spell out in Assumption 1 below and in Assumption 2 in Section 3.1.

**Assumption 1 (Ownership)** Every member of the alliance operates at least one leg in the network, that is, $\mathcal{M}_k \neq \emptyset$ for all $k \in \mathcal{K}$. Moreover, for $i \in \mathcal{M}_k$ there exists $j \in \mathcal{N}_k^l$ such that $\mathcal{L}_j = \{i\}$. Furthermore, each airline collects all the revenues of its local network, that is, $\beta_{kj} = 1$ for every $j \in \mathcal{N}_k^l$.

Assumption 1 imposes rather minimal restrictions on the structure of the alliance, and it is usually satisfied in practice. The specific requirement that each airline operates at least one single-leg itinerary will prove useful in our discussion about optimal proration rates and their connection to some methods currently used in practice to set $\beta$. We note that the condition that each airline

\(^5\)For instance, if leg 4 is the lowest labeled resource owned by airline $k$, the first coordinate of the capacity vector $C_k$ would be $C_k(4)$. 8
collects all the revenues from its local network does not rule out the use of codesharing since code share itineraries belong to the interline network in the set $N^I$. One important implication of this condition is that it guarantees that there is a unique (in a sense that we make precise in the following section) revenue-sharing proration scheme $\beta$ under which the decentralized alliance can achieve full efficiency, that is, the decentralized equilibrium implements the central planner solution.

We model the operations of the alliance using a two-stage hierarchical framework. In the first stage, the members of the alliance negotiate the terms of the agreement used to split the revenues generated by interline and codeshare itineraries. We will restrict our attention to revenue-sharing contracts defined by a fixed proration rule $\beta \in \mathcal{B}$. We will not study the bargaining process among the airlines that produces these proration rates. Instead, we will focus on the outcome of that negotiation process, and explore how the choice of the $\beta$ affects the operation of each airline and the overall efficiency of the alliance in the second stage. Our choice of fixed proration rules (as opposed to rules that would change dynamically the revenue split of interline itineraries) is motivated by the fact that they are prevalently used in practice since they are easy to implement.

In the second stage, given the proration vector $\beta = (\beta_{kj})$, each airline manages its reservation system so as to maximize its expected total revenue during the planning horizon. We model the interaction between the members of the alliance as a non-cooperative game in a decentralized network. In this second stage, each airline must decide how to allocate its capacity among the different products. We restrict our model to the family of SBL. That is, we assume that airline $k$’s strategy is described by two non-negative vectors $y_k = (y_{kj}, j \in N^L_k)$ and $z_k = (z_{kj}, j \in N^I_k)$, where $y_{kj}$ ($z_{kj}$) is the number of seats that airline $k$ allocates to itinerary $j$ in its local (interline) network.

The booking limits form a partition in the sense that the capacity of each leg is partitioned among the different itineraries that use that leg, that is,

$$A^L_k y_k + A^I_k z_k = C_k, \quad \text{for all } k \in K.$$

Note that we are implicitly assuming that for each leg operated by airline $k$, there is enough (local) demand such that the corresponding capacity is saturated. This detail could be relaxed though at the expense of more technical work.

In practice, airlines allocate their capacity dynamically as time goes by and demand information gradually unfolds. There are, however, a couple of reasons that justify our choice of SBL to model airlines’ admission control strategies. First of all, there is the issue of mathematical tractability of modeling and computing dynamic capacity control policies. Even in the monopolistic case, network RM is a challenging problem for which only heuristic solutions (such as bid price or virtual nesting controls) have been successfully proposed. In our setting with multiple players, there is an extra layer of complexity associated to modeling the non-cooperative nature of the game since a dynamic game-theoretical model would at least require including the flow information and beliefs that each airline has about other airlines’ remaining capacities, demand forecasts, and future strategies.

Another reason that justifies (at least in part) our choice of static control relates to the fact that SBL are asymptotically optimal. Specifically, in Section 4, we extend the analysis in Cooper (2002) to show that static booking limits are optimal as the size of the alliance (demand levels and leg
capacities) grows large. Our analysis also reveals that the trade-off between static versus dynamic capacity controls has only a second-order effect on the performance of the alliance. It is the choice of the proration rates $\beta_{kj}$ what really defines this performance. In this respect, our main goal is to characterize the optimal rates $\beta_{kj}$, hence the use of SBL is viewed only as a convenient capacity control mechanism to reach this goal.

Let us turn to the formulation of the optimization problem faced by each airline. Using standard game theory notation, we let $z_{-k} = (z_{kj}, j \in N_k^i, k \in K - \{k\})$ be the set of booking limits selected by the other airlines operating at least one itinerary in $N_k^i$. We denote by $\Pi_k(y_k, z_k, z_{-k}; \beta)$ the expected revenue of airline $k$ as a function of its local and interline strategies $y_k$ and $z_k$, other airlines’ strategies $z_{-k}$, and revenue sharing rule $\beta$,

$$\Pi_k(y_k, z_k, z_{-k}; \beta) = \mathbb{E}\left[\sum_{j \in N_k^i} p_j \min\{D_j, y_{kj}\} + \sum_{j \in N_k^i} \beta_{kj} p_j \min\{D_j, z_{kj}, \min_{k' \in K - \{k\}} \{z_{kj}\}\}\right].$$

Airline $k$’s total payoff is the sum of the expected revenues collected from each itinerary. The term $\beta_{kj} p_j$ is the per seat revenue that airline $k$ gets from interline itinerary $j$. The terms $\min\{D_j, y_{kj}\}$ and $\min\{D_j, z_{kj}, \min_{k' \in K - \{k\}} \{z_{kj}\}\}$ are the number of seats sold in a local or interline itinerary $j$, respectively. These interline sales are the minimum between the demand $D_j$ and the minimum number of seats allocated to itinerary $j$ by those airlines operating this itinerary, i.e., $\min\{z_{kj}, \min_{k' \in K - \{k\}} \{z_{kj}\}\}$. Note that our model follows the traditional independent demand assumption for network RM problems; namely, the demands for different products are mutually independent. An important implication is that it is possible to separate the revenues that each airline gets from its local and interline itineraries. Indeed, once an airline has partitioned the capacity that it will allocate to each network, the operations and revenues of these two businesses decouple.

Our technical analysis consists of two parts. We start by characterizing the set of Nash equilibria of the airline alliance game given a set of proration rates $\beta$, that is, we seek a description of the set $\mathcal{E}(\beta)$ of booking limit vectors $y^* = (y_1^*, y_2^*, \ldots, y_K^*)$ and $z^* = (z_1^*, z_2^*, \ldots, z_K^*)$ given by

$$(y^*, z^*) \in \mathcal{E}(\beta) \iff \forall k \in K, (y_k^*, z_k^*) \in \arg \max_{y_k \geq 0, z_k \geq 0} \Pi_k(y_k, z_k, z_{-k}; \beta)$$

subject to

$$A^1_k y_k + A^1_k z_k = C_k. \tag{1}$$

We are not imposing integrality constraints on the seat allocations. Also, note that a Nash equilibrium $(y^*, z^*)$, as defined in (1), is parametrized by the proration rates $\beta_{kj}$ and we will use the notation $y^*(\beta)$ and $z^*(\beta)$ to emphasize this dependence whenever necessary.

Next, we measure the effects of $\beta$ on the performance of the alliance and, if possible, identify an “optimal” vector of proration rates $\beta^*$. However, what conditions define an optimal $\beta^*$ is subject to discussion as the payoffs of each member of the alliance is affected in a different way by $\beta$ or by the specific equilibrium $(y^*(\beta), z^*(\beta)) \in \mathcal{E}(\beta)$ that one considers. For example, for any $\beta \in \mathcal{B}$, one can show that there exists a Nash equilibrium for which $z_{kj}^* = 0$ for all interline and codeshare itineraries, and for which the payoffs $\Pi_k(z_k^*, z_{kj}^*, z_{-k}^*; \beta)$ are independent of $\beta$. Hence, for this family

$^6$Recall that from Assumption 1, each airline gets the full revenue $p_j$ from local itineraries.
of equilibria the choice of $\beta$ is immaterial. However, the set $E(\beta)$ is rich enough to allow for a non-trivial analysis of the effects of $\beta$ on the performance of the alliance. In particular, we can first attempt to characterize the set of Pareto optimal Nash equilibria for a fixed $\beta$, and then study the impact of $\beta$ on this efficient set of equilibria. In this process, we will pay special attention to one member of this family, namely, the equilibrium that maximizes the aggregate payoff of the alliance. This ‘socially’ optimal equilibrium will allow us to unambiguously define $\beta^*$ as follows,

$$\beta^* \in \arg\max_{\beta \in \mathcal{B}} \max_{(y^*, z^*) \in E(\beta)} \sum_{k \in \mathcal{K}} \Pi_k(y_k^*(\beta), z_k^*(\beta), z_{-k}^*(\beta); \beta).$$

(2)

3. Analysis: Two-stage hierarchical approach

As mentioned above, we proceed backward by first analyzing the operational game given a set of $\beta$, and then the contractual phase that solves (2).

3.1 Stage 2: Operational game

We find convenient to first briefly discuss the sub-problem that each airline faces when optimizing the allocation of capacity to products in its local network. Then, we solve problem (1), characterizing the set of equilibria $E(\beta)$ for a fixed $\beta$ and its subset of Pareto optimal equilibria.

3.1.1 Capacity control in local networks

Let us start by isolating a member $k$ of the alliance. Let $C^i_k, C^u_k \leq C_k$, be the amount of capacity that airline $k$ devotes to itineraries spanning only its local network. Denote by $\Pi_k^L(C_k^i)$ the revenue obtainable from its local network as a function of $C_k^i$. Note that Assumption 1 implies that $\Pi_k^L$ does not depend directly on $\beta$. It follows that

$$\Pi_k^L(C_k^i) := \max_{y_k \geq 0} \sum_{j \in \mathcal{N}_L^k} p_j E[\min\{D_j, y_{kj}\}], \text{ subject to } A_k^L y_k = C_k^L, \quad \theta^L y_k = 0, \quad y_k \geq 0, \quad \theta^L \geq 0,$$

(3)

The optimization in (3) is known as the Probabilistic Nonlinear Programming Problem (PNLP), in this case associated to airline $k$’s local network. It is not hard to see that PNLP is a concave optimization problem with a separable objective function. An optimal solution, denoted by $y(C_k^i)$, can be found solving the following first-order Karush-Kuhn-Tucker (KKT) conditions\footnote{No additional regularity conditions (e.g., Slater conditions) are needed since the constraints in (3) are linear.}:

$$p_j \bar{F}_j(y_{kj}) + \theta_j = \sum_{i \in \mathcal{L}_j} \lambda_i, \quad \forall j \in \mathcal{N}_L^k, \quad A^L_k y_k = C^L_k, \quad \theta \cdot y_k = 0, \quad y_k \geq 0, \quad \theta \geq 0,$$

where $\lambda := \{\lambda_i : i \in \mathcal{M}\}$ are Lagrange multipliers for the capacity constraints and $\theta := \{\theta_j : j \in \mathcal{N}_L^k\}$ are Lagrange multipliers for the nonnegative constraints $y_k \geq 0$. As with the primal variables, we will use the notation $\lambda_i(C^L)$ and $\theta_j(C^L)$ whenever we want to emphasize the dependence of these
dual variables on $C^l$. Note that because there is no interaction among the airlines in the operation of their local itineraries, it follows that $\lambda_i(C^l) = \lambda_i(C^l_k)$ for $i \in M_k$.

For general demand distributions, the solution to the KKT conditions must be computed numerically.\footnote{de Boer et al. (2002) discuss linear approximations to solve PNLP. Chen and Homem-de-Mello (2009) presents an equivalent formulation to PNLP, the SLP (Stochastic Linear Programming Problem), which could be formulated as a two-stage integer programming problem with simple recourse. As noticed in (Talluri and van Ryzin, 2004, Section 3.2.2), when demand is discrete with finite support, one could write the PNLP as a LP albeit one with many more variables. See also Popescu et al. (2005) for alternative methods to solve PNLP.} In the next section, we provide a characterization of the set of Nash equilibria $E(\beta)$ for the airline alliance game that is based on the values of the dual variables $\lambda_i(C^l)$. These are the marginal values (bid prices) of the capacity allocated to the local network. Although the lack of an analytical solution does not affect our theoretical analysis, it does affect its practical implications since our characterization of $E(\beta)$ requires solving a system of equations involving the functions $\lambda_i(C^l)$, where the unknowns appear as a part of the arguments on these functions (see Proposition 1 below). Hence, a (numerical) solution to this system of equations might require evaluating $\lambda_i(C^l)$ for an arbitrary number of values of $C^l$; a task that can be computationally infeasible for large networks. To address this technical nuisance, later we propose a computationally efficient approximation scheme that could be used to estimate the solution of the KKT conditions. We will, however, restrict the generality of the method by imposing the following condition.

Assumption 2 (Local Itineraries) For every local itinerary $j \in N^l$, $F_j(x)$ is strictly increasing for all $x \geq 0$. In addition, in equilibrium, every airline $k$ allocates a positive amount of capacity to each of its local itineraries, that is, $y_k > 0$.

The first part of Assumption 2 (on the monotonicity of $F_j$) is used to simplify the analysis as it implies that the objective function in (3) is strictly concave so that $y(C^l_k)$ is unique. The second part of Assumption 2 is an interior-point condition that will prove useful in our proposed solution method. However, the reader will notice that it is possible to relax this technical requirement at the expense of some additional work. From a practical standpoint, this condition is not particularly restrictive in the sense that we are only requiring that all local itineraries be open at the beginning of the selling horizon.\footnote{Alternatively, we could solve the PNLP (3) and discard the products $j$ with $y_{kj} = 0$. We should caution the reader though because this is with some loss of generality, since when operating the SBL in a dynamic setting, a product that is closed at the beginning might be conveniently open later on when demand unfolds. Nevertheless, in order to deter strategic consumer behavior, in the current airline practice it is common to disclose a commitment to the passengers for which low fare products that are typically available early in the booking horizon, when they become closed, they remain closed from then onwards.}

Under Assumption 2, it follows that $\theta = 0$ and the KKT conditions reduce to the system of equations

$$p_j F_j(y_{kj}) = \sum_{i \in L_j} \lambda_i \quad \text{and} \quad A^l_k y_k = C^l_k.$$  

We can further simplify these conditions by solving for the primal variables $y_{kj}$ in the first equality as a function of the dual variables $\lambda_i$ and then plugging this solution into the capacity constraints.

\footnote{de Boer et al. (2002) discuss linear approximations to solve PNLP. Chen and Homem-de-Mello (2009) presents an equivalent formulation to PNLP, the SLP (Stochastic Linear Programming Problem), which could be formulated as a two-stage integer programming problem with simple recourse. As noticed in (Talluri and van Ryzin, 2004, Section 3.2.2), when demand is discrete with finite support, one could write the PNLP as a LP albeit one with many more variables. See also Popescu et al. (2005) for alternative methods to solve PNLP.}
After some simple manipulations, we get the following nonlinear system of \(|\mathcal{M}|\) equations on \(|\mathcal{M}|\) unknowns, \(\{\lambda_i, i \in \mathcal{M}\}\):

\[
\sum_{j \in N^L_k} A^L_k(i, j) \bar{F}^{-1}_j \left( \frac{1}{p_j} \sum_{s \in L_j} \lambda_s \right) = C^L_k(i), \quad i \in \mathcal{M}_k. \tag{4}
\]

Two cases in which this system admits a simple solution are:

(a) **Single-Leg Itineraries:** Suppose every airline operates a unique, local, single-leg itinerary on each resource, that is, there is a one-to-one mapping between \(N^L_k\) and \(\mathcal{M}_k\) so that \(L_j = \{i\}\) for all \(j \in N^L_k\). In this case, the solution to (4) is trivial and given by \(\lambda_i = p_i \bar{F}_i(C_k(i))\), where \(i\) denotes both a leg in \(\mathcal{M}_k\) and the corresponding single-leg itinerary.

(b) **Uniform Demand Distributions:** Suppose \(D_j\) has a uniform distribution with nonnegative support \([a^L_j - b^L_j, a^L_j]\) for two positive constants \(a^L_j\) and \(b^L_j\). It follows that \(\bar{F}^{-1}_j(x) = a^L_j - b^L_j x\) for \(x \in [0, 1]\). We note that this distribution does not satisfy the first condition in Assumption 2. However, in this particular case we can guarantee the existence of a unique solution \(\mathbf{y} > 0\) by requiring that the support of \(D_j\) is sufficiently large with respect to the available capacity, specifically, we require \(A^L a^L \geq C^L\).

Define the diagonal matrix \(\mathbf{X}^L\) with component \(\mathbf{X}^L(j, j) = b^L_j (p_j)^{-1}\) for \(j \in N^L\) (note that \(k\) is uniquely defined once the local itinerary \(j\) is specified). It follows that the solution to (4) is given by

\[
\lambda = (A^L \mathbf{X}^L (A^L)^\prime)^{-1} (A^L a^L - C^L). \tag{5}
\]

The existence of the inverse matrix \((A^L \mathbf{X}^L (A^L)^\prime)^{-1}\) is guaranteed by the conditions in Assumption 1 as long as the diagonal matrix \(\mathbf{X}^L\) is invertible. Indeed, under these conditions it is not hard to show that \(A^L \mathbf{X}^L (A^L)^\prime\) is a symmetric positive-definite matrix for which an inverse always exists. The nonnegative of \(\lambda\) follows from the condition \(A^L a^L \geq C^L\).

In the context of airline alliances, Case (a) is arguably of limited interest since it is common practice in this industry to offer a variety of different products (booking classes) in the same single-leg route. However, the conditions could be more appropriate in another setting such as a decentralized assembly system in which resources have a more monolithic usage. The conditions in Case (b) are also restrictive, however, they lead to a computationally efficient method to estimate the bid prices (5). One possible way to take advantage of this efficiency without significantly compromising our ability to model demand is to approximate \(D_j\) using a collection \(\{\tilde{D}_{lj} : l = 1, \ldots, L\}\) of uniformly distributed demands. A sketch of this approximation could be found in Section A1 in the Appendix.

### 3.1.2 Nash equilibria for fixed proration rates

In this section we characterize the set of Nash equilibria \((\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{E}(\beta)\) for a fixed value of \(\beta\). To this end, we take advantage of the special structure of the airline alliance game to derive an alternative formulation that will prove useful in solving equation (1). The key step in this reformulation is to
note that $z_{-k}$ impacts the payoff of airline $k$ only through the minimum number of seats that the other airlines are collectively allocating to each interline itinerary. That is, given $z_{-k}$, airline $k$ is only concerned with the meet $\min\{z_{kj} \mid \hat{k} \in K_j - \{k\}\}$ when deciding the number of seats to allocate to itinerary $j \in \mathcal{N}_k^i$.

To capture this special feature of the game, we introduce an auxiliary vector $w = (w_j, j \in \mathcal{N}^i)$. Recall that $\mathcal{N}^i$ is the set of all interline itineraries operated by the alliance. We also define for each airline $k$ the sub-vector $w_k = (w_j, j \in \mathcal{N}_k^i)$. For the purpose of solving the alliance game, we will interpret $w_j$ as a public signal announced by a coordinator\footnote{Although this coordinator is just a fictitious entity that we introduce to explain our mathematical construct, major airline alliances have a central authority or managing partner responsible for coordinating products and services offered by its members.} that specifies a quota on the maximum number of seats that each airline should allocate to itinerary $j \in \mathcal{N}^i$. The advantage of introducing $w$ is that it allows to decompose (1) into a series of simple sub-problems. In particular, we can rewrite airline $k$’s payoff as follows

$$
\Pi_k(C_k, w, \beta) := \max_{y_k \geq 0, z_k \geq 0} \sum_{j \in \mathcal{N}_k^i} p_j E[\min\{D_j, y_{kj}\}] + \sum_{j \in \mathcal{N}_j^i} \beta_{kj} p_j E[\min\{D_j, z_{kj}\}]
$$

subject to

$$
A_I^k y_k + A_L^k z_k = C_k \quad (6)
$$

$$
z_{kj} \leq w_j \quad \text{for all } j \in \mathcal{N}_k^i.
$$

Formulation (6) is similar to the RHS of (1) for a given $k$, except for the fact that the impact of other airlines’ decisions on $\Pi_k$ is now implicitly captured by the set of constraints involving the public signal $w$. Namely, in equilibrium an airline will never allocate more seats to an interline itinerary than the number of seats allocated by the other airlines operating the itinerary. It is also worth noticing that the optimization problem (6) is an extension of the probabilistic nonlinear program (PNLP) in (3) to the airline alliance game and preserves many of its properties. In particular, (6) is also a concave optimization problem with a separable objective function.

The solution to (6), referred as $z_k(w, \beta)$ for its interline allocation, is the best response of airline $k$ to the public signal $w$ and provides an alternative characterization of the set of Nash equilibria for the airline alliance game. To see this, we introduce the following set

$$
\mathcal{W}(\beta) := \{w \geq 0 : \text{In the solution to (6), } z_{kj}(w, \beta) = w_j, \text{ for all } j \in \mathcal{N}^i \text{ and } k \in K_j\}.
$$

Intuitively, for $w \in \mathcal{W}(\beta)$, each airline follows the public signal allocating exactly $w_j$ seats to those interline itineraries in which it operates. The following lemma shows that $\mathcal{W}(\beta)$ indeed characterizes the set of all Nash equilibria.

**Lemma 1** For any $w \in \mathcal{W}(\beta)$, there exists a unique Nash equilibrium $(y_w, z_w) \in \mathcal{E}(\beta)$ such that $z_w = w$ and $y_w = y(C - A^I w)$. Conversely, for any $(y, z) \in \mathcal{E}(\beta)$ there exists a unique $w_{y, z} \in \mathcal{W}(\beta)$ such that $w_{y, z} = z$.

Hereafter we use $\mathcal{W}(\beta)$ and $\mathcal{E}(\beta)$ interchangeably to denote the set of Nash equilibria.
Using the revenue function for the local network, $\Pi_k^L(C_k)$ in (3), we can rewrite the optimization in (6) in terms of the interline booking limits $z_k$ exclusively:

$$\Pi_k(C_k, w, \beta) := \max_{z_k \geq 0} \sum_{j \in \mathcal{N}_k^i} \beta_{kj} p_j \mathbb{E}[\min\{D_j, z_{kj}\}] + \Pi_k^L(C_k - A_k^i z_k)$$

subject to

$$A_k^i z_k \leq C_k \quad (8)$$

$$z_{kj} \leq w_j \text{ for all } j \in \mathcal{N}_k^j.$$ 

Since $z_k = 0$ is a feasible solution, it follows that $\Pi_k(C_k, w, \beta) \geq \Pi_k^L(C_k)$ for all $w, \beta$ and $k$. That is, each airline is willing (in a weak sense) to participate in the alliance independently of the proration rates $\beta$ that are used to split interline and codeshare revenues.

Using the KKT optimality conditions and the interior-point condition in Assumption 2, we can provide an analytical characterization of $W(\beta)$.

**Proposition 1** Under the conditions in Assumption 2, the set of Nash equilibria $W(\beta)$ is the set of all $w \geq 0$ such that $A^i w \leq C$ and

$$\beta_{kj} p_j \bar{F}_j(w_j) \geq \mathbb{I}(w_j > 0) \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^i w_k) \text{ for all } j \in \mathcal{N}_k^i \text{ and } k \in \mathcal{K}. \quad (9)$$

The left-hand side, $\beta_{kj} p_j \bar{F}_j(w_j)$, for $z_{kj} = w_j$, is the additional revenue that airline $k$ gets by marginally increasing the allocation of capacity to the interline itinerary $j$. The right-hand side represents the opportunity cost of the capacity used by itinerary $j$ if it were used locally by airline $k$. Hence, in a Nash equilibrium characterized by $w$, no airline has an incentive to deviate from the current allocation and transfer capacity from its interline network to any of its local itineraries. An alternative interpretation is to consider (9) as participation constraints that must be verified by a recommendation $w$ to be an equilibrium. The following are some general properties of $W(\beta)$ that follow from the characterization in (9).

**Corollary 1**

1. $W(\beta)$ is a bounded meet-semilattice\(^{11}\), that is, if $w \in W(\beta)$ and $0 \leq w' \leq w$ then $w' \in W(\beta)$.

2. Either $W(\beta) = \{0\}$ or $W(\beta)$ contains infinitely many Nash equilibria.

3. If $w, w' \in W(\beta)$ and $w' \leq w$, then $w$ Pareto dominates $w'$ in the sense that the payoff of each airline under $w$ is greater than or equal to the payoff under $w'$.

The first property implies that if a certain level of collaboration, measured by the allocation $w$, is a Nash equilibrium, then any (component-wise) lower level of collaboration can also be sustained as a Nash equilibrium. The second property follows directly from the first one and establishes that either the alliance is not sustainable in equilibrium (i.e., all interline itineraries are closed) or there

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11Consider a partially ordered set $(S, \preceq)$. We say that $S$ is a meet-semilattice if $\forall x, y \in S$, the greatest lower bound of the set $\{x, y\}$ is in $S$. 

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are infinitely many different ways in which the airline can collaborate. The final property asserts that in the case of multiple equilibria, higher levels of collaboration generate higher revenues to each alliance member. In this respect, the airline alliance game exhibits strategic complementarity.

To further discuss the implications of these properties, let us consider in more detail a simple numerical example in which demands are uniformly distributed. Section A2 in the Appendix contains the derivation of the set $W(\beta)$.

**Example: Uniform demand distributions.** Consider an alliance with two airlines, each controlling a single leg. The alliance offers four itineraries; itinerary $i$ is a local itinerary for airline $i$, $i = 1, 2$, while itineraries 3 and 4 are interline and use one unit of capacity on each leg. The alliance network is schematically depicted in Figure 1. The details about demand distributions, capacities and prices are summarized in Table 1.

![Figure 1](image.png)

Figure 1: An alliance with two airlines, each controlling one leg (Airline 1 controls leg A-B and Airline 2 controls leg B-C) and four itineraries. Itineraries 1 and 2 are local, while itineraries 3 and 4 are interline each using both legs.

<table>
<thead>
<tr>
<th>Revenue per Seat ($\beta_k p_j$)</th>
<th>Capacity $C_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Itinerary 1</td>
<td>Itinerary 2</td>
</tr>
<tr>
<td>$200$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$300$</td>
</tr>
<tr>
<td>Demand Distribution</td>
<td>$U[100,500]$</td>
</tr>
</tbody>
</table>

Table 1: Network data for the example in Figure 1. In this example, the price vector is $p = (200, 300, 350, 350)$, and the proration rates are $\beta_1 = (1, 0, 5/7, 2/7)$ and $\beta_2 = (0, 1, 2/7, 5/7)$ for Airlines 1 and 2, respectively.

Note that despite the fact that itineraries 3 and 4 consume the same set of resources, they generate different revenues for the two airlines; Airline 1 gets $250 per seat sold in itinerary 3 and only $100 per seat sold in itinerary 4. This could be an example in which itinerary 3 and itinerary 4 are sold through the separate selling channels of Airline 1 and Airline 2, respectively.

In this example, the public signal is a two-dimensional vector $w = (w_3, w_4)$, each component representing the allocation of capacity to each of the interline itineraries. The left panel in Figure 2 depicts the set of Nash equilibria $W(\beta)$. This set takes the form of a polyhedron in the positive orthant when
demands are uniformly distributed (see Section A2 in the Appendix for details). In this case, the active constraints (edges) of \( W(\beta) \) are given by the non-negativity constraints on \( w \) (segments A-B and A-D), Airline 1’s participation constraint on itinerary 4 (segment B-C)\(^{12}\) and Airline 2’s participation constraint on itinerary 3 (segment C-D). The right panel in Figure 2 shows the set of achievable payoffs for the two airlines for the set of Nash equilibria \( W(\beta) \). There is a one-to-one mapping between the regions in both panels. It is worth noticing that the region of payoffs is not convex for the set of pure-strategy Nash equilibria \( W(\beta) \), hence, it is possible to (strictly) enlarge this region using randomization.

In terms of efficiency, the non-collaboration equilibrium in point A serves as a benchmark to evaluate the value of the alliance. This value depends, of course, on the specific equilibrium under consideration but it is clear that both airlines are better off engaging in some degree of collaboration. Pareto efficiency is reached in the upper-right boundary B-C-D, this follows more generally from condition 3 in Corollary 1. Point C is the most efficient equilibrium in the set \( W(\beta) \) in the sense that the sum of the payoffs of the two airlines \( (\Pi_1 + \Pi_2) \) is maximized at C. We note, however, that Point C does not characterize the first-best allocation of capacity from the alliance perspective, which is achieved under a centralized control (we will return to this point in the next section). If we compare the payoffs of the two airlines in the extreme cases, Point A (no collaboration) and Point C (full collaboration) we conclude that collaboration increases the payoff of the alliance in approximately 50%. For the individual airlines, collaboration increases payoffs for Airlines 1 and 2 by 21% and 95%, respectively. □.

The discussion in the previous example unveils a number challenges that need to be addressed in the coordination of the alliance, most notoriously the fact that the game has multiple equilibria leading to a wide range of outcomes. One thing that seems certain is that the members of the alliance should look for ways to reach an equilibrium that lies on the Pareto frontier\(^{13}\). To address

\(^{12}\)In other words, in the boundary B-C, Airline 1 is indifferent between interchanging capacity between itinerary 4 and its local itinerary 1; that is, equation (9) holds at equality for \( w \) over that segment.

\(^{13}\)On the other hand, the question of which specific equilibrium should be selected on this frontier has a less obvious answer and depends on the relative bargaining power of the different airlines.
this issue, in Section A4 of the Appendix we propose a simple mechanism in which the alliance coordinator iteratively collects and feedbacks information from and to the airlines that will help them reach an efficient equilibrium. Interestingly, the only transfer of information from the airlines to the coordinator is the sign of the local bid prices. In other words, airlines are not required to share detailed demand information. In addition, the fact that this information about bid prices is centralized by the coordinator and is never openly revealed provides another layer of privacy.

3.2 Stage 1: Contract design

In this section we explore the characteristics of a proration rule under which the performance of the alliance is maximized (see equation (2) for a definition). We also compare the best performance in the decentralized system to the one achieved under centralized control, and propose a heuristic proration rule implementable just based on publicly available fares.

3.2.1 Central planner solution

We start by defining the centralized problem, which yields an upper bound on the alliance performance. In a centralized scenario, a central planner chooses a partitioned allocation to all itineraries, \((y^C, z^C)\), where the superscript ‘C’ stands for Centralized. Although in a centralized system there is no distinction between local and interline itineraries, we will retain this terminology and the notation of the previous section to help the exposition. In particular, \(y^C_j (z^C_j)\) is the optimal centralized booking limit for an itinerary \(j \in \mathcal{N}^L\) (\(j \in \mathcal{N}^I\)). The central planner’s problem is given by

\[
\Pi^C(C) := \max_{y^C \geq 0, z^C \geq 0} \sum_{j \in \mathcal{N}^L} p_j \mathbb{E}[\min\{D_j, y^C_j\}] + \sum_{j \in \mathcal{N}^I} p_j \mathbb{E}[\min\{D_j, z^C_j\}]
\]

subject to \(A^L y^C + A^I z^C = C\). (10)

Problem (10) is yet another version of PNLP for which the KKT conditions fully characterize an optimal solution. As with the local itineraries at the airline level, we will simplify the analysis by assuming that the interior-point condition in Assumption 2 holds for the centralized system, that is \(y^C > 0\). As a result, we have a simple characterization for the bid prices of the single-leg, local itineraries. For any leg \(i \in \mathcal{M}\)

\[
\lambda_i^C(y^C) := p_j \bar{F}_j(y^C_j), \quad \text{for any } j \in \mathcal{N}^L \text{ such that } L_j = \{i\}.
\]

The assumption \(y^C > 0\) guarantees that we can define the bid price for leg \(i\) based on the marginal value of capacity for any of the single-leg local itineraries that uses that leg.\(^{14}\) The next result will prove useful in the sequel and follows from the KKT conditions for problem (10).

\(^{14}\)We can slightly relax the assumption \(y^C > 0\), and allow \(y^C \geq 0\) as long as for any \(j \in \mathcal{N}^L\), there exists a leg \(i \in L_j\) and \(L_j = \{i\}\), where \(j \in \mathcal{N}^L\), such that \(y^C_j > 0\). In words, we are requiring that for each itinerary \(j\), there exists a leg \(i \in L_j\), such that the central planner would allocate some positive capacity to at least one single-leg itinerary that only consists of leg \(i\). In that case, the definition of \(\lambda_i^C(y^C)\) is adjusted by

\[
\lambda_i^C(y^C) = \max\{p_j \bar{F}_j(y^C_j) : j \in \mathcal{N}^L, L_j = \{i\}\}.
\]
Lemma 2 Suppose that $\mathbf{y}_i^C > 0$. If $z_j^C > 0$ for some $j \in \mathcal{N}^t$, then $p_j \tilde{F}_j(z_j^C) = \sum_{i \in \mathcal{L}_j} \lambda_i^C(\mathbf{y}_i^C)$.

To understand the lemma, note that $p_j \tilde{F}_j(z_j^C)$ is the marginal revenue from interline itinerary $j$ and $\sum_{i \in \mathcal{L}_j} \lambda_i^C(\mathbf{y}_i^C)$ is the sum of the marginal revenues from the corresponding single-leg local itineraries used by leg $j$. So the condition asserts that in any centralized optimal solution the marginal revenue of any interline itinerary should be equal to the marginal opportunity cost of reallocating capacity to local single-leg itineraries. Otherwise, the central planner can do better by reallocating space from interline itinerary $j$ to the local itineraries $i \in \mathcal{L}_j$.

3.2.2 Optimal proration rule

Turning to the efficiency of a decentralized solution, for a given $\beta$ and a given vector of capacity allocation $(C_1, C_2, \ldots, C_K)$, the alliance maximum aggregate payoff is given by

$$\Pi^C(C_1, C_2, \ldots, C_K, \beta) := \sup_{\mathbf{w} \in \mathcal{W}(\beta)} \sum_{k \in \mathcal{K}} \Pi_k(C_k, \mathbf{w}, \beta). \quad (12)$$

It follows from their definitions that $\Pi^C$ is uniformly bounded above by $\Pi^D$. A natural question, at this point, is whether $\Pi^C(C_1, C_2, \ldots, C_K, \beta^*) := \max_{\beta \in \mathcal{B}} \Pi^D(C_1, C_2, \ldots, C_K, \beta)$ can match the central planner payoff $\Pi^C(C)$. In the following result, we provide a positive answer to this question under rather general conditions. Furthermore, we explicitly derive the optimal proration rule $\beta^*$ in terms of the centralized solution $(\mathbf{y}_i^C, z^C)$. An evident, yet key, observation in this derivation is that we only need to guarantee that the central planner’s allocation of capacity to the interline itineraries, $z^C$, can be sustained in the decentralized system. Indeed, once the allocation of capacity to these interline itineraries has been decided, the optimization problem of how to distribute the remaining capacity among the local itineraries is the same for the centralized and decentralized systems.

Theorem 1 Suppose that $\mathbf{y}^C > 0$. Consider the subset of revenue-sharing rates $\mathcal{B}^*(\mathbf{y}_i^C, z^C) \subseteq \mathcal{B}$ defined as follows: $\beta^* \in \mathcal{B}^*(\mathbf{y}_i^C, z^C)$ if and only if $\beta^*_{kj} = 1$ for all $j \in \mathcal{N}_k^t$ and

$$\beta^*_{kj} = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^C(\mathbf{y}_i^C)}{\sum_{i \in \mathcal{L}_j} \lambda_i^C(\mathbf{y}_i^C)} \quad \text{for all } j \in \mathcal{N}_1^t \text{ such that } z_j^C > 0. \quad (13)$$

Then, $\mathcal{B}^*(\mathbf{y}_i^C, z^C)$ contains all the revenue-sharing proration rates, $\beta^*$, under which the centralized optimal allocation $(\mathbf{y}_i^C, z^C)$ is a Nash equilibrium in the decentralized system, and the decentralized alliance achieves the centralized payoff, i.e., $\Pi^D(C_1, C_2, \ldots, C_K, \beta^*) = \Pi^C(C)$. In other words, $z^C \in \mathcal{W}(\beta^*)$ if and only if $\beta^* \in \mathcal{B}^*(\mathbf{y}_i^C, z^C)$. Moreover, no subset of players want to deviate collusively from this equilibrium strategy.

A few comments about this result and implications are in order. First, if an itinerary $j \in \mathcal{N}_1^t$ is not profitable for the alliance as a whole (i.e., $z_j^C = 0$), then there is no need to impose any constraint on how to allocate the revenues of this itinerary among the airlines ($\beta_{kj}$ can be arbitrary chosen). This follows, trivially, from the fact that the allocation $z_j = 0$ can always be implemented in a Nash
equilibrium. On the other hand, if itinerary \( j \) is profitable for the central planner (i.e., \( z^C_j > 0 \)) then \( \beta^*_k \) is set in such a way that each airline in the decentralized system has enough incentives to allocate the optimal number of seats to itinerary \( j \) in equilibrium. These incentives are determined using exclusively the central planner’s allocation of capacity to the local itineraries so that each airline gets exactly the proportion of the opportunity cost that it contributes to the itinerary.

These optimal proration rates have some direct practical implications on the type of agreements that the members of the alliances should be trying to achieve.

**Corollary 2** Under the assumptions in Theorem 1:

1. Proration rates should be defined at the route level\(^{15} \) and not at the product level. That is, \( \beta^*_{kj} = \beta^*_{kj'} \) for any pair of itineraries \( j \) and \( j' \) that cover the same route (\( L_j = L_{j'} \)) and which are both open in the centralized solution (\( z^C_j > 0 \) and \( z^C_{j'} > 0 \)).

2. Airlines should collect no revenues on those itineraries they do not (partially) operate, that is, if \( z^C_j > 0 \) and \( L_j \cap M_k = \emptyset \) then \( \beta^*_{kj} = 0 \).\(^{16} \)

3. There is a unique way to choose the proration rates \( \beta^*_{kj} \) for those interline itineraries \( j \) for which \( z^C_j > 0 \) so that the first-best efficient solution (\( y^C, z^C \)) can be implemented in a decentralized system.

Typically the number of routes in the network is significantly smaller than the number of itineraries offered by the alliance, so the first point can prove very useful in practice. The second point is intuitive from an economic perspective but reduces the incentive for codesharing. Indeed, it asserts that marketing airlines that sell products that they do not operate should get no compensation for their services. Regarding the ‘uniqueness’ property of \( \beta^* \) in the third point, we note that this condition relies on our requirement that each airline collects all the revenues generated by its local network (posed in Assumption 1). Indeed, in the absence of this condition there are, possibly, a continuum of sharing rules that can implement the centralized solution. To see this, consider the following family of perfect sharing rules defined by \( \beta_{kj} = \beta_k \) for all itineraries \( j \in N \), where \((\beta_1, \beta_2, \ldots, \beta_K)\) is a \( K \)-dimensional vector such that \( \sum_k \beta_k = 1 \) and \( \beta_k \geq 0 \) for all \( k \in K \). Under this perfect sharing rule, airline \( k \) gets a fixed proportion \( \hat{\beta}_k \) of the revenues generated by any itinerary in the entire network. As a result, the expected payoff of each airline is a fixed proportion of the payoff of the aggregate system, and so the centralized solution (\( y^C, z^C \)) can be trivially implemented\(^{17} \).

Despite the fact that these perfect sharing rules can coordinate the alliance, they appear to be of limited practical value. Indeed, in order to implement these rules and their associated payoff

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\(^{15}\)Recall that a route is a sequence of flight legs, potentially involving several fare-class combinations.

\(^{16}\)Following-up with our Iberia-American case in Section 2, even though our model allows Iberia to sell a ticket in the American operated, single-leg itinerary JFK-MAD, it will not get any compensation under optimal proration rates.

\(^{17}\)We note that to define feasible proration rates, the vector \((\beta_1, \beta_2, \ldots, \beta_K)\) must satisfy the following participation constraint (i.e., each airline has to collect at least the same revenues that they can collect if they stay out of the alliance) \( \hat{\beta}_k \Pi^C(C) \geq \Pi^C_k(C_k) \) for all \( k \in K \). From the inequality \( \Pi^C(C) \geq \sum_k \Pi^C_k(C_k) \) we conclude that there always exists a perfect sharing rule that satisfies the participation constraint (e.g., \( \hat{\beta}_k = \Pi^C_k(C_k)/\Pi^C(C) \)). Moreover, if the inequality is strict there are infinitely many.
functions, each airline would have to monitor the sales process of every single itinerary on the entire network. On the other hand, for a given set of proration rates $\beta^* \in \mathcal{B}^*(y^C, z^C)$ defined in Theorem 1, each airline only needs to monitor the sales process of those itineraries in which it operates (point 2 in Corollary 2), a feature that certainly favors its implementation. It is important to notice, however, that even though any $\beta^* \in \mathcal{B}^*(y^C, z^C)$ can implement the centralized solution without requiring information sharing among the airlines, the actual computation of the set $\mathcal{B}^*(y^C, z^C)$ depends on the value of the centralized solution $(y^C, z^C)$ and corresponding bid prices $\lambda_i^C(y^C)$. So unless a central authority (such as the alliance managing partner) can compute this solution and airlines are willing to share private information $D_j$ about local itineraries, it can be difficult for the members of the alliance to agree on a vector of proration rates that belongs to $\mathcal{B}^*(y^C, z^C)$.

In the following section, we explore further this problem and propose a simple mechanism that produces a vector of proration rates that is close (and in some cases belongs) to the set $\mathcal{B}^*(y^C, z^C)$ and that at the same time does not require airlines sharing private information or computing the centralized solution.

### 3.2.3 A nearly optimal proration rule

As we discussed in the previous section, independently of the revenue-sharing rule that is used, airlines are always better off joining the alliance than staying out of it. At the same time, each individual airline would like to implement a proration rule that maximizes its own revenue. This tension can possibly lead to a complicated negotiation process, one for which a formal analysis is beyond the scope of this paper. Instead, we will propose a simple proration rate rule that relies on observable data, and that airlines can use to reach a solution that is close to an efficient solution in the sense of Theorem 1.

Let us first discuss the similarities between the proration rates in Theorem 1 and one of the alternatives that is commonly used in practice, namely, proration based on local fares. Under this type of proration, the fraction of the fare that an operating airline gets from an interline itinerary is proportional to the fares of the single-leg itineraries. To be more precise, consider an interline itinerary $j \in \mathcal{N}^3$ and let $p_i$ be the fare of a single-leg itinerary $i \in \mathcal{L}_j$, then the proration rates based on local fares are computed as follows

$$
\beta_{kj}^f := \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} p_i}{\sum_{i \in \mathcal{L}_j} p_i}.
$$

(14)

Note that $\beta_{kj}^f$ may not be well defined since it is possible to have multiple single-leg itineraries using leg $i$ and so the fare $p_i$ is not uniquely specified (e.g., different fare classes on the same flight). In practice, this ambiguity is resolved establishing a mapping from the set of interline itineraries to the set of single-leg itineraries based on booking classes with similar services and restrictions. For example, fare class Y in an interline itinerary may be mapped to fare class Y in the corresponding single-leg itineraries.\(^\text{18}\)

\(^{18}\)It is common practice in the airline industry to use capital letters to denote booking classes that impose similar

21
From the definition of the bid prices in equation (11) and the result in Theorem 1, we can see the resemblance between the proration rule $\beta^*$ and the one defined by local fares $\beta^f$. The following result formalizes this connection.

**Proposition 2** Consider an interline itinerary $j \in \mathcal{N}$ such that $z_j^C > 0$, and suppose that for each of the legs $i \in \mathcal{L}_j$ there exists a single-leg local itinerary $\tilde{\mathcal{Y}}_i$ such that the fill rates of these local itineraries are all equal in the centralized solution, that is, there exists a constant $\alpha_j$ such that $\bar{F}_i(y_{\tilde{\mathcal{Y}}_i}^C) = \alpha_j$ for all $i \in \mathcal{L}_j$. Suppose $\beta^f_{kj}$ is computed using the price of these single-leg itineraries

$$\beta^f_{kj} := \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} p_{\tilde{\mathcal{Y}}_i}}{\sum_{i \in \mathcal{L}_j} p_{\tilde{\mathcal{Y}}_i}}.$$

Then, $\beta^*_{kj} = \beta^f_{kj}$.

The condition on the fill rates required in the proposition is rather restrictive and unlikely to be met in practice. However, the previous result suggests a simple heuristic that the members of the alliance can use to approximate $\beta^*$. Indeed, according to this proposition, it would be enough to identify a set of single-leg local itineraries with similar fill rates, for instance products with a small fill rate. Usually, within the set of booking fares associated to a particular route, those with lower fares (and more restrictive conditions) are generally closed first, a fact suggesting that their fill rates are consistently low. This strategy of closing low-fare classes first is also consistent with the low-before-high pattern of demand typically observed in the airline industry (as in other Class A services in terms of Desiraju and Shugan, 1999; see also Talluri and van Ryzin, 2004, chapter 2). The following heuristic proposes a simple mechanism to compute the proration rates and is motivated by this feature of low fare products.

**A Simple Revenue-Sharing Rule:** For each leg $i \in \mathcal{M}$, let the operating airline identify the corresponding single-leg itinerary using leg $i$ with the lowest fill rate. Using a slight abuse of notation, let us define this itinerary by $\tilde{\mathcal{Y}}_i$. Then, for any itinerary $j \in \mathcal{N}'$, define the proration rates

$$\beta^h_{kj} := \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} p_{\tilde{\mathcal{Y}}_i}}{\sum_{i \in \mathcal{L}_j} p_{\tilde{\mathcal{Y}}_i}}. \quad \square$$

It is worth noticing that the computation of $\beta^h$ is distribution free. Furthermore, it only requires as input the mapping $\{\tilde{\mathcal{Y}}_i : i \in \mathcal{M}\}$ and since fares are publicly available, airlines (and the alliance coordinator) can monitor that reported prices $p_{\tilde{\mathcal{Y}}_i}$ coincide with those associated to lowest-fare classes.

The following result provides a uniform bound for the estimation error of this revenue-sharing rule.

restrictions on a ticket like advance purchase requirement, minimum length of stay, change of date fees, etc. For example, class Y usually refers to 'Full Fare unrestricted Economy class'.
Proposition 3 For each leg $i$, let $\delta_i := \bar{F}_{\bar{\eta}_i}(y_{C_{\tilde{j}}})$ so that $1 - \delta_i$ is the fill rate for local itinerary $\tilde{j}_i$ under a centralized solution. Then,

$$\frac{|\beta^*_h - \beta^*_h|}{\beta^*_h} \leq \frac{1}{\delta} - 1$$

for all $j$ such that $z_{c_j}^c > 0$,

where $\delta := \min\{\delta_i : i \in \mathcal{M}\}$.

**Example: Uniform demands (continued).** Consider again the example discussed in Section 3.1.2 (Figure 1 and Table 1). Under the proration rates there, the most efficient Nash equilibrium (Point C in Figure 2) produces payoffs $\Pi_1 = 52,192$ and $\Pi_2 = 47,175$ for airlines 1 and 2, respectively. Using the result in Theorem 1, we can compute the vectors of efficient proration rates $\beta^*_h = (1, 0.495, 0.495)$ and $\beta^*_h = (0, 1, 0.96, 0.96)$, which generate the payoffs $\Pi^*_h = 55,338$ and $\Pi^*_h = 50,540$. Finally, using the proposed revenue-sharing rule, we obtain the proration rates $\beta^*_h = (1, 0.4, 0.4)$ and $\beta^*_h = (0, 1, 0.6, 0.6)$, with associated payoffs $\Pi^*_h = 50,953$ and $\Pi^*_h = 53,395$. The inefficiency of the sharing rule described in Table 1, measured as $1 - (\Pi_1 + \Pi_2)/(\Pi^*_1 + \Pi^*_2)$, is equal to 6.15%. On the other hand, the inefficiency of the proposed rule $\beta^*_h$ is only 1.45%. □

4. **Asymptotic analysis**

In this section we discuss the asymptotic optimality of the fixed proration rates and SBL discussed in the previous sections as the alliance network grows large. This optimality is measured with respect to the case in which airlines take decisions dynamically over time making use of all available information. Our discussion follows closely the analysis in Cooper (2002) for a monopolistic case.

In a dynamic setting, in which airlines observe gradually the evolution of demand and available inventory, capacity decisions and revenue sharing rules need only to be adapted to the available information at the time the decisions are made. Depending on the level of collaboration among the members of the alliance, a particular airline will have access to a (strict) subset of the available history and will have to “fill up the gaps” using its on beliefs about the missing information. The introduction of a system of beliefs usually leads to dynamic games that are notoriously intractable both analytically and computationally. As we mentioned before, this is one of the main reasons to restrict our analysis to static control policies. The obvious concern with this choice is how much the alliance members are “leaving on the table” by implementing suboptimal static policies. As we show in this section, the optimality gap shrinks rapidly as the size of the alliance (in terms of demands and capacities) grows large. To formulate this result, let us denote by $U := \{u = (u_1, \ldots, u_k)\}$ the set of all admissible dynamic policies, where $u_k$ is airline $k$’s strategy, which is a mapping from its available history to capacity control and revenue sharing decisions. Fortunately, we can remain intentionally ambiguous in our description of $U$ since our analysis does not require much details. Let $J_k(u)$ and $J(u) := J_1(u) + \cdots + J_K(u)$ be the expected payoffs that airline $k$ and the entire alliance get, respectively, under policy $u \in U$.

To formalize the notion of asymptotic optimality, we introduce a sequence of problems indexed by a positive scalar $r$, for which both the vector of initial capacities, $C(r)$, and demands, $D(r)$, grow
proportionally large in the value of $r$. Specifically, we assume that
\[
\lim_{r \to \infty} \frac{C(r)}{r} = C \quad \text{and} \quad \lim_{r \to \infty} \frac{D(r)}{r} = \mu,
\]
where $\mu$ is the vector of mean demands and $\frac{\cdot}{\cdot}$ stands for equal in distribution.\(^{19}\) All other parameters are kept independent of $r$. For system $r$, we let $U(r)$ be the set of admissible dynamic policies and $J(u, r)$ be the alliance payoff under an arbitrary dynamic policy $u \in U(r)$. Similarly, $\Pi^D(\beta^*(r), r)$ is the optimal scaled decentralized payoff defined in (12) under the static setting, where $\beta^*(r)$ is an optimal proration rule for system $r$ (defined in Theorem 1). Note that $\sup_{u \in U(r)} J(u, r) \geq \Pi^D(\beta^*(r), r)$ for all $r$.

The following result establishes the asymptotic optimality of the SBL and proration rule.

**Proposition 4** The proration rule $\beta^*(r)$ and associated SBL are asymptotically optimal in the sense that
\[
\lim \sup \sup_{u \in U(r)} \frac{J(u, r)}{\Pi^D(\beta^*(r), r)} = 1.
\]

We conclude this section discussing the incentives that individual airlines might have to deviate from using SBL and switch to an alternative (dynamic) control policy. Our goal here is to measure the robustness and stability of these simple policies when viewed from a Nash equilibrium perspective. For this, let us introduce the notion of an $\epsilon$-Nash equilibrium.

**Definition 1** Consider a stochastic game denoted by $(\Pi, U, K)$, where $\Pi$ represents the vector of payoffs contingent on players’ actions, $U$ is players’ strategy space and $K$ is the set of players. For any $\epsilon \geq 0$, an $\epsilon$-Nash equilibrium is a policy $u^* \in U$, such that no player can gain more than an $\epsilon$-fraction of the optimal payoffs by choosing a different strategy, given that all the other players follow $u^*$. That is, $u^*$ is called an $\epsilon$-Nash equilibrium if
\[
\frac{\Pi_k(u_k, u^*_{-k}) - \Pi_k(u_k^*, u^*_{-k})}{\Pi_k(u_k^*, u^*_{-k})} \leq \epsilon \quad \text{for any } (u_k, u^*_{-k}) \in U \quad \text{and} \quad k \in K.
\]

The following result shows that deviating unilaterally will not bring a significant increase in payoff to an airline in an asymptotic sense.

**Proposition 5** Suppose that $D_j(r)$ has mean $r \mu_j$ and standard deviation $\sigma_j(r) = O(\sqrt{r \mu_j})$, $j = 1, \ldots, n$. Let $(y^*(r), z^*(r)) \in E(\beta^*(r))$ be the optimal SBL for system $r$. Then, $(y^*(r), z^*(r))$ is an asymptotic $\epsilon$-Nash equilibrium, that is, for any $\epsilon \geq 0$, there exists an integer $N_\epsilon$ such that for any $r \geq N_\epsilon$, $(y^*(r), z^*(r))$ is an $\epsilon$-Nash equilibrium. Moreover, for any $r$, the relative improvement $\epsilon_r \leq O(\frac{1}{\sqrt{r}})$.

\(^{19}\)The assumption on the weak convergence of $D(r)/r$ could seem restrictive, but it is satisfied for a large class of examples. For example, the condition is satisfied if $D_j(r)$ has a Poisson distribution with mean $\lambda_j r$ (see Cooper, 2002 for further discussion about this condition).
We note that the condition imposed on the standard deviation of $D_j(r)$ is satisfied by the Poisson distribution, and more generally, by any demand $D_j(r)$ which is the sum of $r$ iid random variables with mean $\mu_j$ and finite variance.

Our results in this section suggest that the use of SBL could lead to a very good performance; not only the total revenue is asymptotically optimal, but also these simple strategies form a set of asymptotic Nash equilibria. These conclusions, however, rely heavily on the use of an optimal revenue sharing rule ($\beta^*$). Hence, an important managerial implication of this analysis is that airlines when forming alliances should put special attention to the process they use to define these revenue sharing rules. Furthermore, the fact that SBL perform well suggests that the issue of how to optimally (and dynamically) allocate capacity among the different products has only a second-order effect on revenues. In the following section we conduct a set of numerical experiments to further discuss these observations.

5. Numerical experiments

We consider two network examples in our numerical study: A hub-and-spoke topology with four airlines, and a general hub-and-spoke network. In the appendix, we have included a section that contains all the data that we use in these experiments. Under each network structure, we study the performance of the entire alliance under various control methods, assuming the revenue share rule follows a fixed proration.

Our numerical study covers SBL, and four other dynamic policies briefly described below (a more detailed specification could be found in Section A5.1 in the Appendix). These benchmark policies are ‘dynamic’ in the sense that the booking horizon is divided into a number of sub-periods (ten in our case) and the capacity allocation decisions are revised at the beginning of each of these time windows.

**Dynamic partitioned booking limits #1 (DBL1):** To study the impact of information (or the lack of it), we consider policy DBL1 under which airlines make inventory allocation decisions without taking into account the impact of the partners’. At each re-solving point, each airline solves its own SBL to maximize its expected future revenue, assuming partners' allocation will never be a constraint. Each airline solves the previous optimization problem and implements the resulting SBL until the next re-solving point.

**Dynamic partitioned booking limits #2 (DBL2):** As opposed to DBL1, this second heuristic assumes inventory information is shared among partners during the booking horizon. In particular, at each re-solving point, each airline updates its SBL based on its remaining inventory, taking into account the partners' inventory. The updated booking limits should form a Nash equilibrium in static sense, and they are used until the next re-solving point.

**Dynamic bid-prices #1 (DBP1):** In a bid-price control policy, an airline estimates an opportunity cost for each itinerary and accepts a request for it if and only if the revenue from the itinerary exceeds
its opportunity cost. We consider a heuristic dynamic bid-price control policy assuming there is no information exchange between the airlines except for accept/reject decisions. Under DBP1, airlines compute a bid price for each leg at the beginning of each time window without taking into account partners’ actions, as if any request for a seat on partners’ legs will always be accepted.\textsuperscript{20}

\textbf{Dynamic bid-prices #2 (DBP2):} As opposed to DBP1, we assume in this case that inventory information is shared among airlines over time. Under DBP2, airlines compute the opportunity cost of the requested itinerary taking into account partners’ actions.

### 5.1 Hub-and-spoke network with four airlines

In this section, we consider the network with five nodes shown in Figure 3 and the data from Williamson (1992) (included for completeness in Section A5.2 in the Appendix). This topology represents a typical hub-and-spoke network with both local and interline itineraries, which is very common and representative in practice. We assume that there are four airlines, each controlling two legs (round-way) on each route. There are four itineraries (or booking classes) on each possible route. This gives a total of 80 possible itineraries: 32 local and 48 interline. We assume the demand for each itinerary is a Poisson process. The mileage data that we use in the experiment is real world data between the airports shown in Figure 3.

![Four airline network](image)

**Figure 3:** Four airline network (Williamson (1992)). Each airline controls a pair of legs from/to the hub.

#### 5.1.1 Alternative prorations rates

First, we assume all airlines use SBL to control capacity decisions. We focus on measuring the performance of the alliance under alternative proration rules, $\beta$, in particular we consider three alternatives: our proposed optimal solution, $\beta^*$, computed in Theorem 1, heuristic prorations based on local fares, $\beta^h$, computed in (14) and prorations based on mileage, $\beta^m$ (under which an airline gets a fraction of the fare of an interline that is equal to the proportion of the total mileage of the itinerary that it operates). Figure 4 plots the performance of the alliance under these three revenue

\textsuperscript{20}We have chosen this particular heuristic because some commercial airlines are currently using it to manage their capacity.
sharing rules as a function of the demand factor, defined as the overall demand divided by total capacity, i.e.,

\[
\text{Demand factor} := \frac{\sum_{j \in N} \sum_{i \in \mathcal{L}} \mu_j}{\sum_{i \in \mathcal{M}} c_i},
\]

where \( \mu_j \) is the mean demand of itinerary \( j \) and \( c_i \) is the capacity of leg \( i \). The demand in each itinerary is assumed to be a Poisson process.

Figure 4: Alliance cumulative revenue for three revenue sharing rules (\( \beta^*, \beta^m, \beta^h \)) as a function of the demand factor. The curves \( \Pi^*, \Pi^m \) and \( \Pi^h \) represent the performance under the optimal, heuristic and mileage-based prorations, respectively.

We can see that the optimal proration \( \beta^* \) dominates the mileage-based revenue sharing proration and the heuristic rates. In particular, the mileage-based proration can lead to a revenue loss of 25\% with respect to the optimum. While mileage-based prorations can lead to a poor solution, the heuristic proration rates tend to perform better (a fact that we have observed across all our computational experiments). This empirical observation together with the analytical result in Section 3.2 suggest that the conditions identified in Proposition 2 might (approximately) hold in practice and so using our proposed heuristic to set the proration rates can lead to a simple and yet effective mechanism to coordinate the alliance.

5.1.2 Dynamic vs. static control policies under an optimal revenue sharing rule

In the second part, we study the alliance performance when the various control methods described above are used. We assume that proration rates are kept fixed and equal to \( \beta^* \) during the entire booking horizon, and compute the performance of SBL, DBL1, DBL2, DBP1 and DBP2. In addition, we test the performance of SBL under the heuristic proration rates \( \beta^h \). For an arbitrary policy \( \mathcal{H} \), let \( J^\mathcal{H}(r) \) be the performance of the alliance under this policy in a system of size scaled by \( r \) (for example, \( J^{\text{DBP1}}(r) \) is the cumulative revenue under policy DBP1). For each policy, we compute its relative performance with respect to the upper bound \( \Pi^{CE}(r) \) (see equation (A2) in the Appendix).
as follows:

\[ \mathcal{P}^H(r) := \frac{J^H(r)}{\Pi^CE(r)}. \]

### Table 2: Relative performance of alternative capacity control policies

<table>
<thead>
<tr>
<th>Asymptotic Regime Index (r)</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{P}^{SBL})</td>
<td>0.881</td>
<td>0.902</td>
<td>0.916</td>
<td>0.925</td>
<td>0.940</td>
<td>0.948</td>
</tr>
<tr>
<td>(\mathcal{P}^{SBLh})</td>
<td>0.798</td>
<td>0.818</td>
<td>0.833</td>
<td>0.843</td>
<td>0.854</td>
<td>0.861</td>
</tr>
<tr>
<td>(\mathcal{P}^{DBL1})</td>
<td>0.715</td>
<td>0.685</td>
<td>0.683</td>
<td>0.694</td>
<td>0.679</td>
<td>0.679</td>
</tr>
<tr>
<td>(\mathcal{P}^{DBL2})</td>
<td>0.916</td>
<td>0.970</td>
<td>0.939</td>
<td>0.958</td>
<td>0.968</td>
<td>0.981</td>
</tr>
<tr>
<td>(\mathcal{P}^{DBP1})</td>
<td>0.678</td>
<td>0.694</td>
<td>0.696</td>
<td>0.688</td>
<td>0.684</td>
<td>0.666</td>
</tr>
<tr>
<td>(\mathcal{P}^{DBP2})</td>
<td>0.930</td>
<td>0.971</td>
<td>0.951</td>
<td>0.984</td>
<td>0.986</td>
<td>0.987</td>
</tr>
</tbody>
</table>

Table 2 presents the relative performance as a function of the asymptotic regime index \(r\). Regarding the performance of policies under optimal proration rates, bid price control with full information (DBP2) appears to dominate the other alternatives. However, SBL works very well, reaching a performance above 92.5% of the upper bound for values \(r \geq 1\). This again supports our claim that dynamic inventory control may be a secondary issue under an optimal revenue sharing rule.

Three additional insights can be derived from Table 2. First, one can show that the difference in performance between dynamic policies using the same information structure is less than 3% (that is, comparing DBL1 vs. DBP1, or DBL2 vs. DBP2). This indicates that the performance of the alliance is approximately invariant with respect to the choice of dynamic inventory control policies that use the same level of information under the optimal proration rates \(\beta^*\). Second, under booking limit control and bid price controls, the value of information exchange for the alliance is close to 30%. Third, SBL under heuristic proration rates delivers at least 8% more revenues than other (dynamic) policies with no information sharing.

### 5.2 General hub-and-spoke networks

In this part of our numerical study we consider the general hub-and-spoke network in Figure 5 using data publicly available from Topaloglu (2007). The purpose of the study is two-fold. First, we investigate the impact of the demand factor (i.e., the ratio of demand to capacity) on the performance of the alliance under alternative inventory control policies when the optimal proration rates \(\beta^*\) are used. Second, we study how the number of airlines in the alliance affects its performance. We assume that there are two round-way flight legs associated with each spoke (inbound and outbound legs) and each airline controls one spoke and the corresponding pair of legs. We assume again that there is a single itinerary on each possible route (we only use the low-fare class from Topaloglu’s dataset). We use \(K\) to denote the number of spokes. Consequently, we have \(|\mathcal{M}| = 2K\) legs and \(|\mathcal{N}| = K(K + 1)\) itineraries of which \(2K\) are single-leg local itineraries and \(K(K - 1)\) are interline itineraries.
5.2.1 Effect of the demand factor on the alliance performance

Figure 6 depicts the normalized alliance revenue, i.e., the alliance cumulative revenue divided by the demand factor as function of the demand factor. When the demand factor is small, revenues are nearly flat and they begin to drop as the demand factor grows large. The intuition is as follows. When demand is small compared to capacity, all requests are accepted and so in this range the normalized revenue increases in proportion to the demand factor as demand increases. As demand keeps increasing capacity will eventually get tight forcing the airlines to reject some requests and so the normalized revenues will only increase at a slower pace than the demand increases. Through our numeric study, we find that the performance of all policies is very close. This is consistent with our findings in the previous subsection.

5.2.2 Effect of the number of airlines on the alliance performance

We study the effect of network size on the alliance revenue. To this end, we compute revenues under SBL controls, and examine various networks with $K = 2, 4, 6, \text{ and } 8$ spokes. In our numeric analysis,
across the four networks, we kept the average capacity of each leg, the average fare of each itinerary, and the demand factor constant. Figure 7 shows that alliance revenue under such assumption has decreasing returns to scale in the number of airlines.

![Figure 7: Alliance performance as a function of the number of spokes.](image)

### 6. Concluding remarks and extensions

In this paper, we propose a framework to study revenue-wise the operation and collaboration of airline alliances in a decentralized network. In particular, we model the revenue sharing of alliances using a two-stage hierarchical framework. In the first stage, alliance members negotiate proration rates that they use to split the revenues generated by their interline and codeshare itineraries. In the second stage, given these proration rates, each airline manages its own capacity in a non-cooperative fashion so as to maximize its expected cumulative revenue for the booking horizon under consideration. In our model, we assume that airlines use static partitioned booking limits (SBL) to control capacity.

Our study of the alliance operational and contractual problem proceeds backward in the hierarchy. First, we characterize the set of Nash equilibria for fixed revenue sharing prorations that occurs in stage 2. Then, in stage 1, we investigate the effect of different revenue sharing rates on the performance of the alliance and identify an optimal rule.

An important feature of this game is the multiplicity of Nash equilibria leading to a wide range of possible outcomes (from no collaboration at all to socially optimal levels of collaboration). The set of equilibria exhibits strategic complementarity in the sense that airlines’ incentives to increase the allocation of capacity to interline itineraries are (weakly) aligned. From a managerial perspective, this feature suggests that airlines could work together to seek coordinating strategies that are Pareto efficient. A simple (tâtonnement type of) mechanism is proposed (details in the Appendix) to help airlines reach an efficient equilibrium. This mechanism relies heavily on the existence of an alliance coordinator capable of acting as an intermediator that canalizes the flow of information among the airlines. In fact, such central coordinator already exists in major alliances.

In studying the effects of alternative revenue sharing prorations, we find that the overall performance of the alliance is very sensitive to the particular sharing rule that is utilized. We identify an optimal revenue sharing proration; one under which the central planner solution belongs to the set of
Nash equilibria of the decentralized game. The computation of these optimal proration rates requires significant level of collaboration among the airlines (in terms of demand information sharing), which is a fact that might prevent their implementation in practice. To address this limitation, we also investigate the connection between these optimal proration rates and a particular type of proration rates which are commonly used in practice, computed using data on local fares. We identify conditions under which these fare-based proration rates coincide with the optimal and use these conditions to devise a simple method to approximate the optimal proration rates based on publicly available data, and that can be used in practice.

By means of asymptotic analysis, we prove that using optimal static proration rates together with static partitioned booking limits is asymptotically optimal and so airlines have limited incentives to use more sophisticated dynamic control systems to manage capacity. We test numerically several dynamic heuristic policies and compare them to static partitioned booking limit controls. Our results show that the choice of a particular revenue sharing rule has a significant impact on the performance of the alliance. Furthermore, the trade-off between static and dynamic capacity controls has only a second-order effect on this performance.

Our model and results can be extended in a number of directions. First of all, in this paper we have not discussed the mechanism (or negotiation process) of stage 1 used by the airlines to determine the revenue sharing rule they will implement. This is a complex process that can be approached from various angles, such as cooperative game or bargaining theory. In this respect, we view our results in this paper as a natural first step which allows to understand how airlines will act and what payoffs they will get once a particular sharing rule has been selected. Another aspect of the model that deserves more attention is our assumption of independent demands. There is a growing interest in the Revenue Management literature to model demand using consumer choice models capturing various type of substitution effects. Bringing these ideas into an airline alliance setting can prove very challenging (for instance, we can no longer decouple local and interline itineraries the way we did in this paper) but we think it is an important extension to pursue. Finally, we mention the issue of modeling dynamic control policies. In this paper, we have focused mainly on static booking limits because of their tractability; however, we were able to show that this assumption is not very restrictive since these simple controls perform noticeably well. In practice, commercial airlines use dynamic control methods and it seems appropriate to investigate how to adapt our methodology to handle this additional feature.

References


A1. Approximation scheme for solving PNLP under general demands $D_j$

In light of our discussion at the end of Section 3.1.1, it is important to notice that we are not interested in any approximation of $D_j$ but one that can preserve the structure of the optimization in (3) and the resulting optimality conditions. In other words, we would like to identify a set of uniform random variables $\{\tilde{D}_{lj} : l = 1, \ldots, L\}$ that satisfies the following condition for all $x$ in the domain of $D_j$

$$\mathbb{E}[\min\{D_j, x\}] \approx \max_{x \geq 0} \sum_{l=1}^{L} \alpha_{lj} \mathbb{E}[\min\{\tilde{D}_{lj}, x_l\}] \quad \text{subject to} \quad \sum_{l=1}^{L} x_l = x.$$  \hfill (A1)

In other words, we are interested in replacing itinerary $j$ by $L$ independent itineraries so that the revenue of allocating $x$ units of capacity to itinerary $j$ is approximately equal to the revenue of reallocating optimally the same $x$ units of capacity among these new itineraries. The parameters $\{\alpha_{lj}\}$ are introduced to improve the quality of the approximation.

Suppose that each $\tilde{D}_{jl}$ has a uniform distribution with support $[0, \delta_{lj}]$. Then, the right-hand side in (A1) can be written as the following quadratic problem

$$H_j(x; \alpha_j, \delta_j) := \max_{x \geq 0} \sum_{l=1}^{L} \left( \alpha_{lj} x_l - \frac{\alpha_{lj}}{2\delta_{lj}} x_l^2 \right) \quad \text{subject to} \quad \sum_{l=1}^{L} x_l = x, \quad x_j \leq \delta_j.$$  \hfill (A2)

The value of the parameters $\alpha_j$ and $\delta_j$ could then be determined by minimizing the error of the approximation in (A1), for instance, minimizing the distance (in $L^2$)

$$R_j^2 := \inf_{\alpha_j, \delta_j} \sup_x \left( \mathbb{E}[\min\{D_j, x\}] - H_j(x; \alpha_j, \delta_j) \right)^2.$$  \hfill (A3)

For an arbitrary demand distribution, one generally needs to compute this approximation for every itinerary $j$. At this point, it is reasonable to wonder if the approximating scheme described above may lead to a more complex computational method to solve (3) than simply tackling the KKT conditions in (4) directly. Let us reiterate that the real benefit of this method has to be found on the ability that it gives us to solve the airline alliance game discussed in the next section rather than to solve the local network subproblem presented in this section. There is one important special case, however, for which the process can be simplified considerably: Suppose that $D_j$ is modeled as
a normal random variable with mean \( \mu_j \) and variance \( \sigma_j^2 \). In this case, it follows (using standard normalization arguments) that it would be enough to compute a good approximation for a single distribution. Indeed, let \( Z \sim N(0,1) \) and let \( H_0(x) \) be a uniform approximation of the form

\[
H_0(x) = \max_{x \geq 0} \sum_{l=1}^{L} \alpha_l \mathbb{E}[\min\{D_l, x\}] \quad \text{subject to} \quad \sum_{l=1}^{L} x_l = x.
\]

Define, the approximation error

\[
R_0^2 := \sup_x \left( \mathbb{E}[\min\{Z,x]\} - H_0(x) \right)^2.
\]

Then, if \( D_j \) is normally distributed with variance \( \sigma_j^2 \), it follows that \( R_j^2 \leq \sigma_j^2 R_0^2 \). Hence, one only needs to put the effort to derive a good uniform approximation for \( Z \).

A2. Calculations for Example 1

Suppose \( D_j \) has a uniform distribution with nonnegative support \( [a_j-b_j, a_j] \) for two positive constants \( a_j \) and \( b_j \), for all \( j \in \mathcal{N} \). It follows that \( F_j^{-1}(x) = a_j - b_j x \) for \( x \in [0,1] \). Let us denote by \( X \) the diagonal matrix with component \( X(j,j) = b_j (\beta_{kj} p_j)^{-1} \) for \( j \in \mathcal{N} \). The diagonal sub-matrices \( X_L \) and \( X^L_k \) are defined from \( X \) using the conventions described in Section 3.1.1. We also define the diagonal matrix \( M_k \) with component \( M_k(i,i) = 1 \) if \( i \in M_k \) for \( k \in \mathcal{K} \).

From the optimality conditions in (9) and the characterization of the dual variables \( \lambda_i \) for uniform demands in (5), we get that the set \( \mathcal{W}(\beta) \) is defined by the following system of linear equations

\[
A^t w \leq C \quad \text{and} \quad 0 \leq w \leq a^t - X^t_k (A^t)^t M_k (A^t X^t (A^t)^t)^{-1} (A^t a^t + A^t w - C) \quad \text{for all} \ k \in \mathcal{K}.
\]

Hence, with uniformly distributed demands, \( \mathcal{W}(\beta) \) is a bounded polytope.

A3. Technical proofs

**Proof of Lemma 1**

Given any vector \( w \in \mathcal{W}(\beta) \), define the strategy \((z_k, y_k)\) for airline \( k \) as follows:

\[
\begin{align*}
    z_k &= (z_{kj} : z_{kj} = w_j, \text{ for all } j \in \mathcal{N}_k); \\
    y_k &= y_k (C_k - A_k^t z_k).
\end{align*}
\]

By definition of \( w \), no player has incentive to deviate from \( z_k \), so the strategy derived from \( w \) forms a Nash equilibrium, i.e., \((z_k, y_k) \in \mathcal{E}(\beta)\).

In the other direction, if \((y, z) \in \mathcal{E}(\beta)\), then \( z_{kj} = z_{k'j} \) for any \( j \in \mathcal{N} \), and for any \( k, k' \in \mathcal{K} \).

The reason is that the actual number of seats sold in itinerary \( j \) will never exceed the smallest
It follows that at optimality allocation of capacity in the local network.

Then, by complementary slackness, for all \( w \), such that \( w = (w_j : w_j = z_{kj}, \text{for } k \in \mathcal{K}_j) \). By definition of Nash equilibrium \( \mathcal{E}(\beta) \), \( z_{kj} \) will solve (6) and therefore \( w \in \mathcal{W}(\beta) \). □

**Proof of Proposition 1**

Under Assumption 2, problem (8) is a (strictly) concave optimization problem and an optimal solution is characterized by the following KKT conditions:

\[
\begin{align*}
\beta_{kj} p_j \bar{F}_j(z_{kj}) + \frac{\partial}{\partial z_{kj}} \Pi_k^i(C_k - A_k^i z_k) &= \eta_j - \theta_j + \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \delta_i, \\
A_k^i z_k &\leq C_k, \quad z_{kj} \leq w_j, \quad 0 \leq z_{kj} \\
\delta_i (A_k^i z_k - C_k) &= 0, \quad \eta_j (z_{kj} - w_j) = 0, \quad z_{kj} \theta_j = 0, \\
\delta_i &\geq 0, \quad \eta_j \geq 0, \quad \theta_j \geq 0.
\end{align*}
\]

It follows that at optimality \( z_{kj} = w_j \) if and only if

\[
\beta_{kj} p_j \bar{F}_j(w_j) = -\frac{\partial}{\partial z_{kj}} \Pi_k^i(C_k - A_k^i z_k) \bigg|_{z_k = w} + \eta_j - \theta_j + \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \delta_i.
\]

Then, by complementary slackness, for all \( w_j > 0 \) we have that \( \theta_j = 0 \). This, together with the nonnegativity of \( \delta_i \) and \( \eta_j \) implies that \( z_{kj} = w_j \) if and only if

\[
\beta_{kj} p_j \bar{F}_j(w_j) = -\frac{\partial}{\partial z_{kj}} \Pi_k^i(C_k - A_k^i z_k) \bigg|_{z_k = w} \quad \text{for all } j \in \mathcal{N}_k \text{ such that } w_j > 0
\]

\[
= \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^i w),
\]

where the equality follows from the fact that \( \lambda_i \) is the shadow price of resource \( i \) with respect to the allocation of capacity in the local network. □

**Proof of Corollary 1**

1. \( \lambda_i(C_i, \beta) \) is the Lagrange multiplier for the capacity constraint on leg \( i \in \mathcal{M} \). Since the objective is a concave function in the capacity \( C_i \), \( \lambda_i(C_i, \beta) \) is a nonincreasing function in the coordinates of \( C_i \). Therefore, if \( w \in \mathcal{W}(\beta) \), i.e., \( \beta_{kj} p_j \bar{F}_j(w_j) \geq \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^i w_k) \), and \( 0 \leq w' \leq w \), then \( \beta_{kj} p_j \bar{F}_j(w_j') \geq \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^i w_k') \).

2. This can be derived directly from 1.

3. From equation (6), the maximization problem with \( w, w \geq w' \), has a larger feasible region and the same objective function compared with the maximization problem with \( w' \). Therefore, \( \Pi_k(C_k, w, \beta) \geq \Pi_k(C_k, w', \beta) \), for \( k \in \mathcal{K} \). □
Proof of Theorem 1

By definition of $\beta^*$, we have for $z_j^C > 0$ that

$$\beta^*_{kj}p_j \bar{F}_j(z_j^C) = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^C(y^C)}{\sum_{i \in \mathcal{L}_j} \lambda_i^C(y^C)} p_j \bar{F}_j(z_j^C), \text{ for } j \in \mathcal{N}_k^I \text{ and } k \in \mathcal{K}.$$  

From Lemma 2 and equation (11), we get

$$\frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^C(y^C)}{\sum_{i \in \mathcal{L}_j} \lambda_i^C(y^C)} p_j \bar{F}_j(z_j^C) = \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^C(y^C).$$

For $i \in \mathcal{M}_k$, $\lambda_i^C(y^C)$ is the central planner’s marginal value of capacity of resource $i$ operated by airline $k$. Furthermore, since $y^C > 0$ we have by equation (11) that

$$\lambda_i^C(y^C) = p_j \bar{F}_j(y_j^C) \text{ for some } j \in \mathcal{N}_k^C \text{ such that } \mathcal{L}_j = \{i\},$$

that is, $\lambda_i^C(y^C)$ is the marginal value of resource $i$ when used in the local network operated by airline $k$. But, the problem of how to optimally allocate a fixed amount of capacity in the local networks is the same under a centralized and decentralized operations (this follows from the fact that $\beta_{kj} = 1$ for all $j \in \mathcal{N}_k^I$). As a result, we have that the centralized and decentralized marginal value of capacity in resource $i$ coincide, that is,

$$\lambda_i(y^C) = \lambda_i(C_k - A_k^1 z_k^C) \text{ for } i \in \mathcal{M}_k.$$

Combining the previous identities we get that

$$\beta^*_{kj}p_j \bar{F}_j(z_j^C) \geq \mathbb{1}(z_j^C > 0) \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^1 z_k^C) \text{ for all } j \in \mathcal{N}_k^I \text{ and } k \in \mathcal{K}.$$  

From Proposition 1, it follows immediately that $z^C$ is a Nash equilibrium under the revenue sharing rule $\beta^*$ in the decentralized system.

Finally, for any subset of airlines, we can consolidate the legs operated by these airlines and treat this set of players as a single airline. The conclusion holds for the same reason as no player wants to deviate unilaterally. □

Proof of Corollary 2

The first two points follows directly from Theorem 1. To prove the uniqueness of $\beta^*$ on those itineraries $j$ with $z_j^C > 0$, suppose there exists another proration rule $\beta_{kj}$ that implements the centralized solution as a Nash equilibrium. Then, it follows from Proposition 1 that

$$\beta_{kj}p_j \bar{F}_j(z_j^C) \geq \mathbb{1}(z_j^C > 0) \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(C_k - A_k^1 z_k^C) \text{ for all } j \in \mathcal{N}_k^I \text{ and } k \in \mathcal{K}.$$

A4
Summing over all $k$ such that $\mathcal{L}_j \cap \mathcal{M}_k \neq \emptyset$ we get that
\[
p_j \tilde{F}_j(z_j^c) \sum_{k: \mathcal{L}_j \cap \mathcal{M}_k \neq \emptyset} \beta_{kj} \geq \mathbb{I}(z_j^c > 0) \sum_{i \in \mathcal{L}_j} \lambda_i(c_k - A_k^i z_j^C) = \mathbb{I}(z_j^c > 0) \sum_{i \in \mathcal{L}_j} \lambda_i^c(y^c).
\]
See the proof of Theorem 1 for a discussion on the last equality. Using the result in Lemma 2 we get that
\[
\sum_{k: \mathcal{L}_j \cap \mathcal{M}_k \neq \emptyset} \beta_{kj} \geq \mathbb{I}(z_j^c > 0).
\]
By the definition of the set of proration rates $\mathcal{B}$, the summation is always less than or equal to one. We conclude that for $z_j^c > 0$ the summation above is exactly equal to one. This conclusion has two consequences. First, we must have $\beta_{kj} = 0$ for all $k$ such that $\mathcal{L}_j \cap \mathcal{M}_k = \emptyset$. Second, all the inequalities above must hold as equalities. In particular, for all $j$ such that $z_j^c > 0$ we have
\[
\beta_{kj} = \sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i(c_k - A_k^i z_j^C) / p_j \tilde{F}_j(z_j^c) = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^c(y^c)}{\sum_{i \in \mathcal{L}_j} \lambda_i^c(y^c)}.
\]

**Proof of Proposition 2**

From the definitions of the bid prices in equation (11) and $\tilde{\alpha}_i$, we have $\lambda_i^c(y^c) = p_i \tilde{F}_j(y_j^c)$, for $i \in \mathcal{L}_j$. Using the assumption that $\bar{\alpha}_i = \alpha_j$ for all $i \in \mathcal{L}_j$, and the result in Theorem 1,
\[
\beta_{kj}^* = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} \lambda_i^c(y^c)}{\sum_{i \in \mathcal{L}_j} \lambda_i^c(y^c)} = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} p_i \tilde{F}_j(y_j^c)}{\sum_{i \in \mathcal{L}_j} p_i \tilde{F}_j(y_j^c)} = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} p_i \tilde{F}_j(y_j^c)}{\sum_{i \in \mathcal{L}_j} p_i} = \beta_{kj}^f.
\]

**Proof of Proposition 3**

Let us fix the indices $k$ and $j$. From the definition of $\beta_{kj}^*$ and $\beta_{kj}^h$, it follows that for all $j$ such that $z_j^c > 0$
\[
\frac{\beta_{kj}^*}{\beta_{kj}^h} = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} w_i \delta_i}{\sum_{i \in \mathcal{L}_j} \theta_i \delta_i},
\]
where
\[
w_i := \frac{p_{i_j}}{\sum_{s \in \mathcal{L}_j \cap \mathcal{M}_k} p_{s_i}} \quad \text{and} \quad \theta_i := \frac{p_{i_j}}{\sum_{s \in \mathcal{L}_j} p_{s_i}}.
\]
It follows that both the numerator and denominator are convex combinations of the $\delta_i$. Since $\delta \leq \delta_i \leq 1$ for all $i \in \mathcal{M}$, we conclude that
\[
\delta \leq \frac{\beta_{kj}^*}{\beta_{kj}^h} = \frac{\sum_{i \in \mathcal{L}_j \cap \mathcal{M}_k} w_i \delta_i}{\sum_{i \in \mathcal{L}_j} \theta_i \delta_i} \leq \frac{1}{\delta}
\]
and the result follows.
Proof of Proposition 4

The first step is to bypass the difficulties of computing an optimal dynamic policy in \( \sup_{u \in \mathcal{U}(r)} J(u, r) \) by computing an upper bound.

The upper bound is obtained from the deterministic version of the central planner’s problem in equation (10), which follows by replacing the random demand \( D(r) \) by its mean \( \mu(r) \). The resulting optimization problem for system \( r \) is given by

\[
\Pi^{CE}(r) := \max_{y^C \geq 0, z^C \geq 0} \sum_{j \in \mathcal{N}^L} p_j \min \{ \mu_j(r), y^C_j \} + \sum_{j \in \mathcal{N}^I} p_j \min \{ \mu_j(r), z^C_j \}
\]

subject to \( A^L y^C + A^I z^C = C(r) \). \hfill (A2)

We denote by \((y^{CE}(r), z^{CE}(r))\) an optimal solution. The following is a well known result in the revenue management literature that completes the first step of our two-step approach (see (Talluri and van Ryzin, 2004, chapter 3) for details).

Lemma A1 \( \Pi^{CE}(r) \) is an upper bound on the alliance cumulative revenue. That is, for any admissible policy \( u \in \mathcal{U}(r) \), \( J(u, r) \leq \Pi^{CE}(r) \).

Proof: Let \( N^L_j(u, r) \) denote the number of sales of local itinerary \( j \), and let \( N^I_j(u, r) \) denote the number of sales of interline itinerary \( j \) under an arbitrary dynamic policy \( u \in \mathcal{U}(r) \). The expected total revenue is

\[
J(u, r) = \sum_{j \in \mathcal{N}^L} p_j E[N^L_j(u, r)] + \sum_{j \in \mathcal{N}^I} p_j E[N^I_j(u, r)],
\]

where \( N_j(u, r) \) satisfies (pathwise) the following constraints:

\[
N_j(u, r) \leq D_j(r), \text{ for } j \in \mathcal{N}, \hfill (A3)
\]

\[
A^L N^L(u, r) + A^I N^I(u, r) \leq C(r),
\]

\[
N(u, r) \geq 0.
\]

Taking expectation on the constraints, we can see that \( y^C_j = E[N^L_j(u, r)] \) and \( z^C_j = E[N^I_j(u, r)] \) are feasible for the deterministic linear program (DLP in terms of the RM literature):

\[
\Pi^{CE}(r) := \max \sum_{j \in \mathcal{N}^L} p_j y^C_j + \sum_{j \in \mathcal{N}^I} p_j z^C_j
\]

subject to \( y^C_j \leq \mu_j(r) \), for \( j \in \mathcal{N}^L \), \hfill (A4)

\[
z^C_j \leq \mu_j(r) \), for \( j \in \mathcal{N}^I \),
\]

\[
A^L y^C + A^I z^C \leq C(r)
\]

\[
y^C \geq 0, z^C \geq 0.
\]

Since the deterministic version of the central planner’s problem in (A2) is equivalent to the linear program (A4), and the analysis holds for an arbitrary policy \( u \), the objective function value \( \Pi^{CE}(r) \) verifies \( \Pi^{CE}(r) \geq J(u, r) \), for all \( u \in \mathcal{U}(r) \). \( \square \)

To complete the proof of Proposition 4, we first need the following convergence result that follows from the asymptotic properties of the \( r \)-th system in (15).
Lemma A2 The certainty equivalent solution \((y^{CE}(r), z^{CE}(r))\) and its corresponding payoff \(\Pi^{CE}(r)\) satisfy
\[
\lim_{r \to \infty} \frac{1}{r} ((y^{CE}(r), z^{CE}(r), \Pi^{CE}(r)) = (y^{CE}, z^{CE}, \Pi^{CE}),
\]
where \((y^{CE}, z^{CE}, \Pi^{CE})\) are the solution and corresponding payoff for DLP in (A2) replacing \(\mu(r)\) and \(C(r)\) by the limits \(\mu\) and \(C\), respectively, in (15).

Proof. The proof is based on the following result:

Lemma A3 Let \(w^{(k)}\) be a sequence of random vectors defined on a probability space \((\Omega, \mathcal{F}, P)\). Assume that the random vector converges in distribution to random vector \(\omega^0\), i.e. \(\lim_{k \to \infty} w^{(k)} \overset{D}{=} \omega^0\). Let \(x^{(k)}\) be the solution to the stochastic program:
\[
\max_{x \in V} E[g(x, \omega^{(k)})]
\]
and let \(x^0\) be the unique solution to the stochastic program:
\[
\max_{x \in V} E[g(x, \omega^0)].
\]
where \(V\) is a compact set and \(g(\cdot)\) is a continuous function. Then \(x^{(k)}\) converges to \(x^0\), i.e. \(\lim_{k \to \infty} x^{(k)} = x^0\).

Proof. Without loss of generality, we just prove for the case of random variables. By definition, we have \(E[g(x^{k}, \omega^k)] \geq E[g(x^0, \omega^0)]\), which implies:
\[
\liminf_{k \to \infty} E[g(x^{k}, \omega^k)] \geq E[g(x^0, \omega^0)].
\]
By contradiction, suppose that \(\lim_{k \to \infty} x^{(k)} = x^0\) does not hold, then there exists a subsequence of \(\{x^{k}\}\), denoted by \(\{x^{ki}\}_i\), where \(\lim_{i \to \infty} x^{ki} = y\), and \(y \neq x^0\). Taking the limit, we get
\[
\lim_{i \to \infty} E[g(x^{ki}, \omega^{ki})] = E[g(y, \omega^0)].
\]
Since \(x^0\) is the unique solution to \(\max_{x \in V} E[g(x, \omega^0)]\), \(E[g(y, \omega^0)] < E[g(x^0, \omega^0)]\). So we have,
\[
\lim_{i \to \infty} E[g(x^{ki}, \omega^{ki})] < E[g(x^0, \omega^0)]
\]
which is a contradiction. Hence, \(\lim_{k \to \infty} x^{(k)} = x^0\). □

Continuing with the proof of Lemma A2, we apply the previous result to get
\[
\lim_{r \to \infty} \frac{1}{r} ((y^{CE}(r), z^{CE}(r)) = (y^{CE}, z^{CE}). \quad (A5)
\]
From (A5) and continuity of the revenue function, we get:
\[
\lim_{r \to \infty} \frac{1}{r} ((\Pi^{CE}(r)) = (\Pi^{CE}). \quad □
\]
Finally, the proof of Proposition 4 follows as a direct application of Lemmas A1 and A2, and Proposition 1 in Cooper (2002). \(\Box\)

**Proof of Proposition 5**

Given other airlines’ strategy \(z^*_k\) in the original dynamic system, the maximal expected revenue airline \(k\) can get is equal to

\[
 J_k(z^*_{-k}) := \sup_{u_k \in U_k} J_k(u_k, z^*_{-k}) = \sup_{u_k \in U_k} \sum_{j \in N^L_k} p_j \mathbb{E}[N^L_j(u_k, z^*_{-k})] + \sum_{j \in N^L_k} \beta_{kj}^* \mu_j \mathbb{E}[N^L_j(u_k, z^*_{-k})]
\]

where \(N^L_j(u_k, z^*_{-k})\) satisfies pathwise

\[
 N_j(u_k, z^*_{-k}) \leq D_j, \quad \text{for } j \in N_k^k, \\
 A^L N^L_k(u_k, z^*_{-k}) + A^L N^L_k(u_k, z^*_{-k}) \leq C_k \\
 N_j(u_k, z^*_{-k}) \leq z_j^*, \quad \text{for } j \in N_k^l, \\
 N_k(u_k, z^*_{-k}) \geq 0.
\]

An upper bound of \(J_k(z^*_{-k})\) is the objective value of the certainty equivalent (CE), deterministic version of the problem:

\[
 \Pi^\text{CE}_k(z^*_{-k}) := \max_{y_k, z_k} \Pi^\text{CE}_k(y_k, z_k, z^*_{-k}) = \max_{y_k, z_k} \left\{ \sum_{j \in N_k^L} p_j y_j + \sum_{j \in N_k^L} \beta_{kj}^* z_j \right\}
\]

subject to

\[
 y_j \leq \mu_j, \quad \text{for } j \in N_k^k, \quad \text{and } z_j \leq \mu_j, \quad \text{for } j \in N_k^l, \\
 A_k^L y_k + A_k^L z_k \leq C_k, \\
 z_{kj} \leq z_j^*, \quad \text{for all } j \in N_k^l, \\
 y_k \geq 0, \quad z_k \geq 0.
\]

(A7)

For the same reason as in Lemma A1, we have \(\Pi^\text{CE}_k(z^*_{-k}) \geq J_k(z^*_{-k})\). Denote the optimal solution in (A7) by \((\bar{y}_k, \bar{z}_k)\). Therefore, we have

\[
 \Pi^\text{CE}_k(z^*_{-k}) = \sum_{j \in N_k^L} p_j \bar{y}_j + \sum_{j \in N_k^L} \beta_{kj}^* \bar{z}_j.
\]

If airline \(k\) chooses the static policy \((y^*_k, z^*_k)\), the expected revenue for airline \(k\) will be \(J_k(y^*_k, z^*_k, z^*_{-k})\), which by definition of \((y^*, z^*)\) is

\[
 \Pi_k(z^*_{-k}, \beta^*) := \max_{y_k, z_k} \left\{ \sum_{j \in N_k^L} p_j \mathbb{E}[\min\{D_j, y_{kj}\}] + \sum_{j \in N_k^l} \beta_{kj}^* p_j \mathbb{E}[\min\{D_j, z_{kj}\}] \right\}
\]

subject to

\[
 A_k^L y_k + A_k^L z_k \leq C_k \\
 z_{kj} \leq z_j^*, \quad \text{for all } j \in N_k^l, \\
 y_k \geq 0, \quad z_k \geq 0.
\]

Since \((\bar{y}_k, \bar{z}_k)\) is feasible for the above optimization problem, we have

\[
 \Pi_k(z^*_{-k}, \beta^*) \geq \sum_{j \in N_k^L} p_j \mathbb{E}[\min\{D_j, \bar{y}_{kj}\}] + \sum_{j \in N_k^l} \beta_{kj}^* p_j \mathbb{E}[\min\{D_j, \bar{z}_{kj}\}] 
\]

(A8)
Now we will find a lower bound of $\mathbb{E}[\min\{D_j, x_j\}]$, using the fact that $\mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]}$. 

$$\mathbb{E}[\min\{D_j, x_j\}] = \mathbb{E}[D_j - (D_j - x_j)^+] = \mu_j - \mathbb{E}(D_j - x_j)^+ = \mu_j - \frac{1}{2}(\mathbb{E}|D_j - x_j| + \mathbb{E}|D_j - x_j|)$$

$$\geq \frac{1}{2}(\mu_j + x_j) - \frac{1}{2}\sqrt{\mathbb{E}[(D_j - x_j)^2]} = \frac{1}{2}(\mu_j + x_j) - \frac{1}{2}\sqrt{\sigma_j^2 + (\mu_j - x_j)^2}$$

$$\geq \frac{1}{2}(\mu_j + x_j) - \frac{1}{2}(\sigma_j + |\mu_j - x_j|).$$

Plugging the lower bound into the RHS of (A8), and using the fact that $\tilde{y}_{kj} \leq \mu_j$ and $\tilde{z}_{kj}$ (see problem (A7)), we have 

$$\Pi_k(z_{-k}^*, \beta^*) \geq \sum_{j \in N_k^1} p_j(\tilde{y}_{kj} - \frac{1}{2}\sigma_j) + \sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{z}_{kj} - \frac{1}{2}\sigma_j).$$

By definition, the maximal relative increase by deviating from the static model is: 

$$\frac{J_k(z_{-k}^*) - \Pi_k(z_{-k}^*, \beta^*)}{\Pi_k(z_{-k}^*, \beta^*)}.$$

Applying the bounds, we have:

$$\frac{J_k(z_{-k}^*) - \Pi_k(z_{-k}^*, \beta^*)}{\Pi_k(z_{-k}^*, \beta^*)} \leq \frac{\Pi_k^{CE}(z_{-k}^*) - \Pi_k(z_{-k}^*, \beta^*)}{\Pi_k(z_{-k}^*, \beta^*)}$$

$$\leq \frac{\sum_{j \in N_k^1} p_j \sigma_j + \sum_{j \in N_k^1} \beta_{kj}^* p_j \sigma_j}{2(\sum_{j \in N_k^1} p_j(\tilde{y}_{kj} - \frac{1}{2}\sigma_j) + \sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{z}_{kj} - \frac{1}{2}\sigma_j))}.$$

In the $r$th system, demand for itinerary $j$, $D_j^{(r)}$, has mean $r\mu_j$, $j = 1, \ldots, n$, and standard deviation $\sigma_j^{(r)} = O(\sqrt{\tau M_j})$. The solution to the corresponding deterministic problem (A7), denoted by $(\tilde{y}_k(r), \tilde{z}_k(r))$, according to the proof of Lemma A2 satisfies: $\lim_{r \to \infty} \frac{1}{\sqrt{\tau}}(\tilde{y}_k(r), \tilde{z}_k(r)) = (\tilde{y}_k, \tilde{z}_k)$. Consequently, the relative increase in the $r$th system is bounded above by

$$\sqrt{\tau}(\sum_{j \in N_k^1} \beta_{kj}^* p_j \sigma_j + \sum_{j \in N_k^1} \beta_{kj}^* p_j \sigma_j)$$

$$\frac{2r(\sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{y}_{kj} - \frac{1}{2}\sigma_j) + \sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{z}_{kj} - \frac{1}{2}\sigma_j))}{2r(\sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{y}_{kj} - \frac{1}{2}\sigma_j) + \sum_{j \in N_k^1} \beta_{kj}^* p_j(\tilde{z}_{kj} - \frac{1}{2}\sigma_j))} = O\left(\frac{1}{\sqrt{\tau}}\right). \quad \square$$

A4. Coordination mechanism to reach an efficient equilibrium

Through the mechanism presented below, the alliance coordinator iteratively collects and feedbacks information from and to the airlines that will help them reach an efficient equilibrium with respect to the allocation of capacity to local and interline itineraries in Stage 2 of our hierarchical approach.

Coordination Mechanism:

- **Step 0. Initialization**: The alliance coordinator announces an initial level of capacity allocation for the interline itineraries $w$ in the interior of $\mathcal{W}(\beta)$. For instance, $w = 0$.

- **Step 1: Bid Price Computation**: Given $w$, each airline $k$ computes $\Pi_k(w, \beta)$ solving (8) and determines the marginal values

$$\nu_{kj} := \frac{\partial}{\partial w_j} \Pi_k(w, \beta), \quad j \in N_k^1.$$

Airline $k$ reports back the vector $\nu_k$ to the alliance coordinator.
**Step 2: Improvement:** For each itinerary $j \in \mathcal{N}$, the coordinator computes

$$\tilde{\nu}_j := \min_{k \in \mathcal{K}_j} \nu_{kj} \quad \text{and} \quad \tilde{\nu}_{\text{max}} := \max_{j \in \mathcal{N}} \tilde{\nu}_j.$$ 

a) If $\tilde{\nu}_{\text{max}} = 0$ then the coordinator stops and announces $w$ as the final level of collaboration.

b) Otherwise, if $\tilde{\nu}_{\text{max}} > 0$ then the coordinator recomputes $w_j$ as follows $w_j \leftarrow w_j + \epsilon \text{sign}(\tilde{\nu}_j)$, for some $\epsilon > 0$ small, and announces this new value of $w$ going back to Step 1. □

The intuition of this mechanism is simple. It tries to greedily improve from suboptimal values of $w$, those for which there exists an interline itinerary for which every airline operating that itinerary would allocate more capacity to it. In this case, the coordinator proposes a new allocation that (marginally) increments these allocations. (In this respect, we can view the mechanism as one that implements a simple steep-ascent method in a decentralized environment). The mechanism stops when for any interline itinerary there exits at least one airline that is no longer willing to increase the capacity allocate to that itinerary.

There are a number of elements in this mechanism that need further analysis and possibly improvements. For instance, the choice of a fixed $\epsilon$ which is independent of the itinerary and independent of the values of the bid prices $\nu_{kj}$ could be fine tuned. Indeed, if one allows for variable and itinerary-dependent values of $\epsilon$ on Step 2 (b), then this additional flexibility could be used to target a better final equilibrium (even possibly the optimal central planer solution). Despite all these limitations, the mechanism offers some degree of parsimony which makes it appealing from a practical perspective.

### A5. Numerical experiments

#### A5.1 Description of the benchmark policies

**DBL1:** At each re-solving point, each airline solves its own SBL to maximize its expected future revenue, assuming partners’ allocation will never be a constraint. Formally, at each resolve point $t$, airline $k$ chooses allocation an $(y_k(t), z_k(t))$ such that

$$\Pi_k(t) := \max_{y_k \geq 0, z_k \geq 0} \sum_{j \in \mathcal{N}_k} \beta_{kj} p_j E[\min\{D_j(t), y_{kj}\}|\mathcal{F}_k(t)] + \sum_{j \in \mathcal{N}_k} p_j E[\min\{D_j(t), z_{kj}\}|\mathcal{F}_k(t)]$$

subject to $A_k^L y_k + A_k^I z_k = C_k(t),$ \hspace{1cm} (A9)

where $C_k(t)$ is the remaining capacity at time $t$, $D_j(t)$ is itinerary $j$’s cumulative demand from time $t$ onwards and $\mathcal{F}_k(t)$ is the history (filtration) of demand and other relevant information available to airline $k$ at time $t$. Each airline solves the previous optimization problem and implements the resulting SBL until the next re-solving point.

**DBL2:** At time $t$, the SBL vectors $y^*(t) = (y_1^*(t), \ldots, y_K^*(t))$ and $z^*(t) = (z_1^*(t), \ldots, z_K^*(t))$ is a Nash equilibrium, that is, they satisfy the condition in equation (1) replacing the capacity and demand vectors by $C(t)$ and $D(t)$, respectively.
**DBP1:** We adopt the *Certainty Equivalent Control* method ([Bertsimas and Popescu (2003)](Bertsimas2003)) to estimate bid prices. Specifically, at every re-solving point \( t \), the following value function is computed:

\[
J_k^{DBP1}(C_k(t), t) := \max_{y_k \geq 0, z_k \geq 0} \sum_{j \in \mathcal{N}_k^L} \beta_{kj} p_j \min\{\mu_j(t), y_{kj}\} + \sum_{j \in \mathcal{N}_k^I} \beta_{kj} p_j \min\{\mu_j(t), z_{kj}\}
\]

subject to \( A_k^T y_k + A_k^T z_k = C_k(t) \),

(A10)

where \( \mu_j(t) \) is the average demand for itinerary \( j \) from time \( t \) onwards.

Under policy DBP1, airline \( k \) computes the opportunity cost of itinerary \( j \in \mathcal{N}_k \) at time \( t \) as

\[
J_k^{DBP1}(C_k(t), t) - J_k^{DBP1}(C_k(t) - A_k(\cdot, j), t), \quad \text{where } A_k(\cdot, j) \text{ is the vector of resources (legs) operated by airline } k \text{ used by itinerary } j.
\]

A request for an itinerary \( j \) received between time \( t \) and the next re-solving point will be accepted if and only if all the operating airlines accept it, that is, if

\[
\beta_{kj} p_j \geq J_k^{DBP1}(C_k(t), t) - J_k^{DBP1}(C_k(t) - A_k(\cdot, j), t) \quad \text{for all } k \in \mathcal{K}_j.
\]

**DBP2:** For this, let us denote by \((y^{DBP1}(t), z^{DBP1}(t))\) the solution to (A10). Using this solution, we will compute a value function \( J_k^{DBP2}(C_k(t)) \) for airline \( k \) in two steps.

**STEP 1:** Compute the booking limits \((y^{DBP2}(t), z^{DBP2}(t))\) as follows. For the interline itineraries, the booking limits satisfy

\[
z_{kj}^{DBP2}(t) = \min_{k' \in \mathcal{K}_j} z_{k'j}^{DBP1}(t), \quad j \in \mathcal{N}^I.
\]

The booking limits for the local itineraries, \(y_k^{DBP2}(t)\), are then computed as the solution to the local network optimization in (3) using the vector of available capacity \( C_k^L(t) = C_k(t) - A_k^T z_k^{DBP2}(t)\) and for the vector of demands \( D(t)\).

**STEP 2:** Compute the value function \( J_k^{DBP2}(C(t)) \) as follows:

\[
J_k^{DBP2}(C(t)) = \sum_{j \in \mathcal{N}_k^L} \beta_{kj} p_j y_{kj}^{DBP2}(t) + \sum_{j \in \mathcal{N}_k^I} \beta_{kj} p_j z_{kj}^{DBP2}(t).
\]

As in the previous case, a request for an itinerary \( j \) received between time \( t \) and the next re-solving point will be accepted if and only if all the operating airlines accept it, that is, if

\[
\beta_{kj} p_j \geq J_k^{DBP2}(C_k(t), t) - J_k^{DBP2}(C_k(t) - A_k(\cdot, j), t) \quad \text{for all } k \in \mathcal{K}_j.
\]

**A5.2 Parameters used in the numerical experiments**

In all the experiments, we have assumed that demand for each itinerary follows independent Poisson processes. The mean (rate) of these demands will be specified for each set of experiments.

For each experiment, we report the data for the case in which the scale of the system is kept at its reference point \( r = 1.0 \). To get the data for the other scaled regimes (such as \( r = 0.4, 0.6, 0.8, 1.0, 1.5, 2.0 \)) one needs to multiply demands and capacities by \( r \).
A5.2.1 Williamson’s network

Table 3 presents the data for Williamson (1992)’s network in Section 5.1. We have assumed that capacities, demands and fares on a reverse itinerary are the same as the original itinerary. For example, the fare for the itinerary Boston to Los Angeles (BOS-LAX) is the same as the fare for the flight Los Angeles to Boston (LAX-BOS) and equal to $575. Similarly, the capacity in the leg Atlanta to Miami (ATL-MIA) is the same as in the leg Miami to Atlanta (MIA-ATL) and is equal to 300 seats.

<table>
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<tr>
<th>O−D market</th>
<th>fare</th>
<th>mean demand</th>
<th>mileage</th>
<th>capacity</th>
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<td>120</td>
<td>945</td>
<td>300</td>
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<td>ATL-LAX/LAX-ATL</td>
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<td>1940</td>
<td>300</td>
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<td>596</td>
<td>300</td>
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<td>300</td>
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