Collision Avoidance in the Air Traffic Management: A Mixed Integer Linear Optimization Approach

A. Alonso-Ayuso, L. F. Escudero, F.J. Martin-Campo

***Abstract*—This paper tackles the collision avoidance problem in the Air Traffic Management Problem (ATM). The problem consists in deciding the best strategy for new aircraft configurations (velocity and altitude changes) such that all conflicts in the airspace are avoided; a conflict being the loss of the minimum safety distance that has to be kept between two aircrafts. A mixed 0-1 linear optimization model based on geometric transformations for collision avoidance between an arbitrary number of aircrafts in the airspace is developed. Knowing initial coordinates, angle direction and level flight, the new configuration for each aircraft is established by minimizing several objective functions like velocity variation and total number of changes (velocity and altitude), and forcing to return to the original flight configuration when no aircrafts are in conflict. Due to the small computational time for the execution, the new configuration approach can be used in real time by using optimization software.**

***Index Terms*—Air traffic management, collision avoidance, mixed integer linear optimization.**

I. Introduction

ATM is currently based on prefixed routes that pilots have to follow according with a certain flight plan. Next years aim is extending the airspace considering "Free Flight", where pilots and airlines can decide freely on the control of the flight, keeping in touch with air traffic controllers. To preserve safety in air flights, the Conflict Resolution Problem has been studied deeply from different points of view.

On a recent paper by EUROCONTROL [1], aimed to specify the required capabilities of Medium-Term Conflict Detection (MTCD) for Air Traffic Management Systems, the MTCD system is required to detect and notify the controller about the probable loss of the required separation between two aircrafts, an aircraft penetrating restricted airspace, or an aircraft blocking airspace that might have been used by some other ones. That paper considers that, although flight data and trajectories are provided to the MTCD, some uncertainty is likely to be on the trajectories. It distinguishes too between tactical and planned trajectories. Kuchar and Yang (2000) [2] and references therein present a survey of conflict, detection and resolution modeling methods with their own classification. Obstacle avoidance using the linearized constrained Uninhabited Aerial Vehicle (UAV) dynamic has been modeled by Richards and How (2002) [3]. According to these authors, the Centralized Model Predictive Control has been widely developed for constrained systems with many results concerning robustness and it has also been applied to the co-operative control of multiple vehicles. By augmenting the system with a binary "target state" indicating whether the target set is reached or not, the authors end up with a hybrid system at hand. Task completion is then guaranteed by imposing a hard terminal equality constraint on the target state. Dell'Olmo and Lulli (2003) [4] describe a model that is solved by using exact optimization software combined with an heuristic approach for large problems. Christodoulou and Costoulakis (2004) [5] propose a Mixed Integer Nonlinear Optimization approach to solve the conflict problem. Their method allows velocity changes and heading angle control to solve all potential conflicts using only standard optimization software, but it may require more computational effort than what it could be affordable. Two integer linear optimization models for conflict avoidance between any number of airborne aircrafts are proposed in [6], the first one a pure 0-1 LP which avoid conflicts by means of altitude changes, and the second one is a mixed 0-1 LP whose strategy is based on velocity and altitude changes where the authors propose them for the medium term Conflict Detection and Resolution.

The main contribution of this paper is to extend and improve the Velocity Changes (VC) model proposed by Pallottino [7] and Pallottino, Feron and Bicchi (2002) [8]. See also [9]. This model considers instantaneous changes. The main extensions to the VC model are as follows: (1) Altitude changes are allowed to avoid infeasible situations in the VC problem caused by the velocity bounds, or "head to head" conflict situations; (2) the case discussed in [8] where the denominator is zero, causing physical collisions between the corresponding aircrafts, is avoided in our proposed model; (3) Different safety radius are considered since the safety radius for an aircraft can be adjusted differently. This feature can be applied to include the wind factor in the model, extending the safety radius if bad weather conditions are existing; (4) A false conflict situation due to the geometric construction is studied; (5) All aircrafts will be forced to return to the initial configuration when the conflict situations are avoided; and (6) All changes in the aircrafts configurations are updated since aircrafts with higher number of changes will be penalized for the equitable distribution of the maneuvers. For this purpose a mixed integer linear optimization (MILO) is proposed. The required computing time for optimizing realistic sets of aircrafts in conflict is so small that the approach can be used in real time operations.

This paper is organized as follows: In Section II the general features of the problem required to build the MILO model are described as well as some changes in the VC model. In section III the VC model is presented. In Section IV the formulation

A. Alonso-Ayuso, L. F. Escudero and F.J. Martin-Campo are with the Department of Statistics and Operations Research, University Rey Juan Carlos, Madrid-Spain. Corresponding author: javier.martin.campo@urjc.es

of our proposed model is developed. Section V presents the full problem formulation as well as the dimensions of the model. Section VI reports the results of the computational experimentation to verify the efficiency of the proposal and its application in real time. Finally, Section VII presents some conclusions and the main lines of future research.

II. Problem Statement

Aerial sectors and a given number F of aircrafts flying in an aerial sector as well as their configurations are considered. An aerial sector being an airspace portion supervised by an air traffic controller (ATC).

Next, all elements concerning velocity and altitude changes model (VAC) are detailed:

Sets

- $\mathcal{F}$, set of flights in the sector $(1, \ldots, F)$.
- $\mathcal{Z}^f$, set of admissible flight levels for aircraft $f \in \mathcal{F}$, $(1, \ldots, Z)$

Parameters

- t, current time.
- w_n, weight (between 0 and 1) for each objective function term for $n = 1, \ldots, 5$

For all $f \in \mathcal{F}$:

- x_f, y_f, the position (abscissa and ordinate) of aircraft f.
- x^1_f, y^1_f, the out of sector position (abscissa and ordinate) of aircraft f.
- t_1, the out of sector predicted time of aircraft f.
- v_f, current velocity in time t_p of aircraft f.
- v^*_f, initial velocity configuration for aircraft f.
- z_f, current flight level in time t_p of aircraft f.
- z^*_f, initial flight level configuration for aircraft f.
- $\underline{v_f}, \overline{v_f}$, minimum and maximum velocity for aircraft f.
- m^*_f, initial direction of motion in $(-\pi, \pi]$ for aircraft f.
- r_f, safety radius for each aircraft f, usually 2.5 nautical miles.
- n^v_f, n^a_f number of changes in velocity and altitude in the sector for aircraft f until the new execution, respectively.
- t^0_f, t^1_f, entrance and exit times for aircraft f in the aerial sector.
- $c^{q^+}_f, c^{q^-}_f$ costs for positive and negative velocity changes for aircraft f.
- c^j_f, cost for number of levels that aircraft f changes.
- c^v_f, c^a_f, costs for changing velocity and altitude for aircraft f.
- $c^{\hat{v}}_f$, cost for different velocity with the initial flight plan for aircraft f.
- $c^{z^*}_f$, cost for different altitude level with the initial flight plan for aircraft f.

Data preprocessing: For all $f \in \mathcal{F}$:

- $\widehat{v_f}$, optimal velocity configuration to arrive to the final sector point at the predicted time for aircraft f.

Data preprocessing: For all $i, j \in \mathcal{F} : i < j$:

- d_{ij}, distance between aircrafts i and j.
- ω_{ij}, is the angle between the line that joins the aircrafts and the abscissa axis. See Fig. 1.
- $\hat{\omega}_{ij}$, angle between aircrafts i and j that depends on the quadrant on which j lies considering aircraft i centered in the origin.
- hth_{ij}, 0-1 parameter that determines if there is a “head to head” conflict for aircrafts i and j.
- sc_{ij}, 0-1 parameter that determines if two aircrafts i and j have the same coordinates x and y.
- pc_{ij}, 0-1 parameter that determines if there is a “pathological case” between aircrafts i and j; see below.
- fc_{ij}, 0-1 parameter that detect if there is a “false conflict” between aircrafts i and j; see below.
- ip_{ij}, intersection point between straight line trajectories for aircrafts i and j.
- d^1_{ij}, distance between the i aircraft position and ip_{ij}. See false conflicts description.
- d^2_{ij}, distance between points $\big(x_i + \cos(m_i), y_i + \sin(m_i)\big)$ and ip_{ij} for aircrafts i and j. See false conflicts description.
- p_{ij}, 0-1 parameter that will be one if the couple of aircrafts i and j will not be taken into account for the conflict resolution. This parameter depends on the criterion decided by the air traffic controller. Notice that this parameter will be one if $fc_{ij} = 1$.

Variables

For all $f \in F$:

- q_f, velocity variation for aircraft f. This variable is real, and we divide it in two nonnegative variables, say, q^+_f and q^-_f, such that $q_f = q^+_f - q^-_f$ as it is standard in optimization, where q^+_f and q^-_f are the positive and negative velocity variation for aircraft f.
- a_f, 0-1 variable that takes value 1 if aircraft f changes its velocity at the end of the current execution and, otherwise, it is zero.
- b_f, 0-1 variable that takes value 1 if aircraft f changes its altitude at the end of the current execution and, otherwise, it is zero.
- ρ_f, nonnegative integer variable that shows the number of levels that the aircraft f ascends or descends.
- β_f, auxiliary nonnegative continuous variable that models the absolute value of the difference between current velocities and initial velocities as a linear function for aircraft f.

For all $f \in F$ and $z \in \mathcal{Z}$:

- ν^z_f, 0-1 variable that takes value 1 if aircraft f is at altitude level z at the end of the current execution and, otherwise, it is zero.

For all $i, j \in \mathcal{F} : i < j$ and $z \in \mathcal{Z}^i \cup \mathcal{Z}^j$ and $n = 1, ..., 5$:

- δ^n_{ijz} auxiliary 0-1 variables to model or-constraint types.

The problem consists in avoiding all conflicts in a certain aerial sector by using a mixed integer linear optimization formulation. Then, some standard optimization software will be used to solve the problem.

III. The VC Problem

In this section the VC model proposed by Pallottino et al. [8] (2002) is briefly presented. Given a set of aircrafts $\mathcal{F}$ with their flight configurations (positions, angle of motion and velocities) the aim is to decide what the best strategy is for the new configuration avoiding all conflicts. The conflict avoidance constraints is based on the geometric constructions in Fig. 1 where all parameters needed for the model are contemplated.

The main idea of this model comes from the construction of the relative velocity vector between two aircrafts i and j that is $\vec{v}_i - \vec{v}_j$, and depending on the tangent of this one and the tangent of angles r and l (see Fig. 1) a conflict situation can be detected. Therefore no conflicts occur between aircrafts i and j if one of the next expressions is satisfied:

$$\frac{(v_i+q_i)\sin(m_i)-(v_j+q_j)\sin(m_j)}{(v_i+q_i)\cos(m_i)-(v_j+q_j)\cos(m_j)} \geqslant \tan(l_{ij}) \quad (1a)$$

$$\frac{(v_i+q_i)\sin(m_i)-(v_j+q_j)\sin(m_j)}{(v_i+q_i)\cos(m_i)-(v_j+q_j)\cos(m_j)} \leqslant \tan(r_{ij}), \quad (1b)$$

where $v+q$ is the optimal velocity for the conflict avoidance. These expressions are nonlinear but if both hands of the inequalities are multiplied by the denominator, the new expression will be linear considering the sign of the denominator since, if it is negative, the sense of the expression must be changed. Then there are four different cases that are modeled by using variables δ^n_{ij}, for $n = 1, 2, 3, 4$. The objective function consists of maximizing the sum of the velocity variations, so all possible conflicts are avoided in minimum time. Next the complete model (including the or-formulation by using δ variables) is presented:

$$\max \sum_{i=1}^{F} q_i = \min \sum_{i=1}^{F} -q_i$$

subject to:

$$\begin{aligned}
(v_i+q_i)\cos(m_i^*)-(v_j+q_j)\cos(m_j^*) &\leqslant M_1(1-\delta^1_{ij})\\
(v_j+q_j)h_j-(v_i+q_i)h_i &\leqslant M_2(1-\delta^1_{ij})\\
(v_i+q_i)\cos(m_i^*)-(v_j+q_j)\cos(m_j^*) &\leqslant M_3(1-\delta^2_{ij})\\
(v_i+q_i)k_i-(v_j+q_j)k_j &\leqslant M_4(1-\delta^2_{ij})\\
(v_j+q_j)\cos(m_j^*)-(v_i+q_i)\cos(m_i^*) &\leqslant M_5(1-\delta^3_{ij})\\
(v_i+q_i)h_i-(v_j+q_j)h_j &\leqslant M_6(1-\delta^3_{ij})\\
(v_j+q_j)\cos(m_j^*)-(v_i+q_i)\cos(m_i^*) &\leqslant M_7(1-\delta^4_{ij})\\
(v_j+q_j)k_j-(v_i+q_i)k_i &\leqslant M_8(1-\delta^4_{ij})\\
\delta^1_{ij}+\delta^2_{ij}+\delta^3_{ij}+\delta^4_{ij} &= 1
\end{aligned}$$

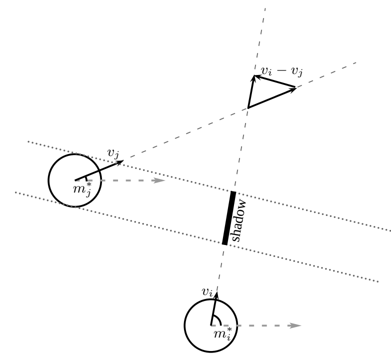
where:

$$\begin{aligned}
h_i &= \tan(l_{ij})\cos(m_i^*)-\sin(m_i^*)\\
h_j &= \tan(l_{ij})\cos(m_j^*)-\sin(m_j^*)\\
k_i &= \tan(r_{ij})\cos(m_i^*)-\sin(m_i^*)\\
k_j &= \tan(r_{ij})\cos(m_j^*)-\sin(m_j^*)
\end{aligned}$$

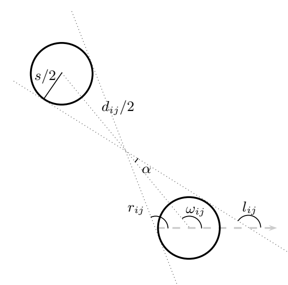
and M_n for $n = 1, \dots, 8$ are large enough numbers.

IV. Conflict Avoidance Constraints for the VAC Problem

A solution to the conflict problem does not always exist in the VC model, since cases such as "head to head", where two different aircrafts are flying along the same straight line with opposite directions, persecution cases where two different aircrafts are flying along the same straight line with the same direction of motion, but one of them is flying faster than the first one, and cases where the bounds of the speeds are insufficient to avoid the conflict situation cannot be solved by only applying velocity changes. Speed changes are also very expensive when the distance between aircrafts is less than 30-40 nmi (see [10]). To avoid these situations, an extension to the VC problem including altitude changes is proposed, resulting in the Velocity and Altitude Changes problem (VAC).



(a) Geometric construction for conflict avoidance constraints



(b) Angles for the conflict avoidance constraints

Fig. 1. Geometric Construction for the VC problem.

The VC problem assumes that all aircraft safety radii have the same value $r_f = s/2$ where s is 5 nautical miles and, under this assumption builds all α angles based on symmetric geometry. Considering different safety radii constitutes a good approximation to the realistic problem, since each aircraft has a different configuration depending on the aircraft weight, the aerodynamic configuration, the aircraft size, etc. When different aircraft radii are considered, two interior tangent lines to two circumferences have to be computed as well as one of these two straight lines slope.

Now, obtaining the α_{ij} angle is easy by using the archtangent of this slope:

$$\alpha_{ij} = \arctan\left(\frac{r_i+r_j}{\sqrt{d_{ij}^2-(r_i+r_j)^2}} \right),$$

where d_{ij} is the distance between the aircrafts i and j. If it is preferred, the α angle can be used to calculate the new l angle or the r angle, see [8], by considering that all safety radii are the same. If the wind factor only acts in one direction the previous is also useful to take into account this factor. This calculation is only valid if the two aircrafts distance is greater than $r_i + r_j$. If the distance is equal to $r_i + r_j$, the slope tends to infinity, since there is only one tangent point. If the distance is less than $r_i + r_j$ the two circumferences have a non empty intersection and only exterior tangent straight lines exist. For these latter cases the parameter sc_{ij} is considered, see below.

With the inclusion of altitude levels, the model of the VC problem has to be expanded, since it can happen that two different aircrafts have a conflict in which a velocity change is insufficient to avoid the conflict situation. However, if the aircrafts fly at different altitude levels, then there will be no conflict. In order to solve these conflict situations, altitude changes will be taken into account and, therefore, some changes in the VC problem will be introduced.

Notice that if altitude changes are considered, the four δ variables in the VC problem have to be modified to include the dimension of the altitude.

Additionally, a new variable δ^5_{ijz} will be included in the model to avoid infeasible situations that could occur in the sum of the δ variables constraint in the VC model. This new variable will take value 0 if there is a conflict between the aircrafts i and j at the same level z; and it will take value 1 if there is no conflict at the same level between those aircrafts. The use of this new variable is advantageous in the sense that it is able to detect infeasible situations given by the velocity bounds when two different aircrafts are nearby or an aircraft flies faster than the other one, for instance.

All VC model constraints must include the dimension z in the δ variables. Also the sum of the δ variables has to be 1. Now, if two different aircrafts fly at different levels, the variable δ^5_{ijz} will be forced to take value 1 and viceversa. Thus, the inclusion of the following constraints achieves our purpose:

$$\nu^z_i + \nu^z_j \geqslant \delta^5_{ijz} - 1 \quad \forall i,j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j$$
$$\nu^z_i + \nu^z_j - 1 \leqslant 1 - \delta^5_{ijz} \quad \forall i,j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j.$$

It is easy to check that if $\delta^5_{ijz} = 1$, then the two aircrafts fly at different levels since $\nu^z_i + \nu^z_j \leqslant 1$ and, if $\delta^5_{ijz} = 0$ the previous sum is greater than -1 and smaller than 2, i.e., the sum is not restricted.

The model approach presented in this paper forces flying aircrafts to lie at different altitude levels in case of a "head to head" conflict. During the first step of the algorithm, preprocessing can detect "head to head" conflicts and, then, a parameter named hth can be fixed to the value 1, while the rest of the hth parameters will be fixed to value 0. All "head to head" cases between every pair of aircrafts can be detected in preprocessing and these cases occur when the following conditions are satisfied (see Fig. 2):

$$\widehat{\omega}_{ij} - \alpha_{ij} \leqslant m^*_i \leqslant \widehat{\omega}_{ij} + \alpha_{ij}$$
$$\widehat{\omega}_{ji} - \alpha_{ji} \leqslant m^*_j \leqslant \widehat{\omega}_{ji} + \alpha_{ji},$$

where $\widehat{\omega}_{ij}$ depends on the quadrant on which (x_j, y_j) lies considering (x_i, y_i) centered in the origin. That is, $\widehat{\omega}_{ij} = \omega_{ij}$ if it is on the first or on the fourth quadrant; $\widehat{\omega}_{ij} = \omega_{ij} + \pi$ if is on the second one; and $\widehat{\omega}_{ij} = \omega_{ij} - \pi$ if is on the third one. Then, in this situation, the parameter hth_{ij} is fixed to 1, forcing the two involved aircrafts to fly at different altitude levels, since previously the δ^5 variables are fixed to 1.

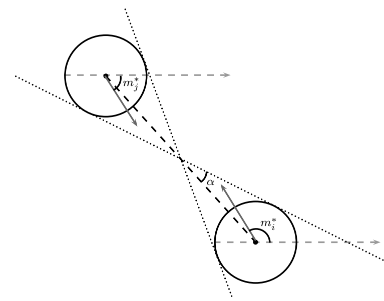


Fig. 2. "Head to head" conflict.

The new altitude configuration will be saved in the ν variables and the following constraint ensures that each aircraft flies at one and only one level:

$$\sum_{z \in \mathcal{Z}^f} \nu^z_f = 1 \quad \forall f \in \mathcal{F}. \tag{4}$$

Now, when a "head to head" conflict occurs, the VAC model forces the two aircrafts in conflict to have different altitude levels as follows:

$$\delta^5_{ijz} = 1 \quad \text{if } hth_{ij} = 1, \forall i,j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j.$$

It can occur that two aircrafts have similar coordinates and they must fly at different altitude levels to avoid collisions. The parameter sc_{ij} will take the value 1 if the distance between two different aircrafts is less or equal to $r_i + r_j$, i.e., the safety distance, and it is zero otherwise. The δ^5 variables can be fixed to 1 in preprocessing as follows:

$$\delta^5_{ijz} = 1 \quad \text{if } sc_{ij} = 1, \forall i,j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j.$$

These previous constraints are used to fix some δ^5 variables to help speeding up the execution but they are not indispensable for the VAC model, since detecting infeasible situations is autonomous. The two previous constraints can be aggregated into one:

$$\delta^5_{ijz} = 1 \qquad \text{if } hth_{ij} + sc_{ij} \geqslant 1, \quad \forall i,j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j.$$

Generally the altitude changes are the cheapest ones in terms of fuel costs, but are the most expensive ones in terms of passenger comfort where horizontal maneuvers are preferred. Also, they increase the safety in maneuvers since the standard vertical separation is 1000 ft opposite horizontal separation that is 5 nmi (see [11], [10]. During the first step of the algorithm, the VAC model could force aircrafts to climb or descend only one level. To avoid possible changes of more than one altitude level the following constraint is used:

$$\nu^z_f \leqslant 0 \quad \forall f \in \mathcal{F}, \forall z \in \mathcal{Z}^f : z \neq z_f, z \neq z_f \pm 1. \tag{5}$$

The constraint (5) may cause infeasible situations if the airspace is very busy and only one change in altitude level is insufficient to solve the problem efficiently. In a busy airspace, it can occur that the VAC model has infeasible situations when aircrafts can not change more than one level. In these cases, we must allow the aircrafts to change two or more levels, but forcing these changes to be as small as possible. For this aim, a new variable say $\rho_f, \forall f \in \mathcal{F}$ will be included in the model. It will take the number of altitude levels that an aircraft climbs or descends. This number of changes can be greater than one and it will avoid infeasible situations in a busy airspace. The number of altitude levels changed by an aircraft can be modeled as follows:

$$\rho_f = \Big| \sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f \Big|.$$

The first addend gives the new level in which the aircraft must be after the optimization, since all ν variables take value 0 except the ν variable in $z \in \mathcal{Z}^f$ in which the aircraft flies after the execution. The second addend gives the initial level in which the aircraft flies before the optimization. The absolute value makes the previous expression positive. Notice that this function is not linear so it has to be transformed. For this purpose, the maximum function between the differences $\sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f$ and $z_f - \sum_{z \in \mathcal{Z}^f} z\nu_f^z$ is taken. That is,

$$\rho_f = \Big| \sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f \Big| = \max\Big\{ \sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f, z_f - \sum_{z \in \mathcal{Z}^f} z\nu_f^z \Big\}.$$

The maximum function is also nonlinear but it can be easily modeled with two additional constraints as follows:

$$\rho_f \geqslant \sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f \qquad \forall f \in \mathcal{F} \qquad (6a)$$

$$\rho_f \geqslant z_f - \sum_{z \in \mathcal{Z}^f} z\nu_f^z \qquad \forall f \in \mathcal{F}. \qquad (6b)$$

The constraints (6) force ρ_f to be the maximum between the two expressions above, because ρ_f must be higher or equal than the two expressions and ρ_f only can be greater or equal than one expression, except there is no level change. In this case, both (6a) and (6b) are trivially fulfilled, ρ_f being zero. Now, all constraints are linear ones and $\sum_{f \in \mathcal{F}} \rho_f$ has to be minimized.

In the VAC model, an objective function consists in minimizing the number of velocity and altitude changes performed by each aircraft. Since the algorithm will be iteratively executed, the updating of the number of changes by each aircraft is necessary to balance the number of maneuvers performed by each aircraft. In a first approach the number of velocity changes will be counted. With constraints (7), it happens that variable a_f takes value 1 if a velocity change occurs and this happens if $|q_f| \neq 0$. The absolute value is a nonlinear function and expressing it as a linear one in the traditional optimization way is required: $|q_f| = q_f^+ + q_f^-$ where $q_f^+ = \max\{q_f, 0\}$ is the positive part of q_f and $q_f^- = \max\{-q_f, 0\}$ is the negative part of q_f. These two new variables are positive and $q_f^+ + q_f^- \neq 0$ is equivalent to $q_f^+ + q_f^- > 0$. Thus,

$$M_1(1 - a_f) + \varepsilon \leqslant q_f^+ + q_f^- \qquad \forall f \in \mathcal{F} \qquad (7a)$$

$$q_f^+ + q_f^- \leqslant M_2 a_f \qquad \forall f \in \mathcal{F}, \qquad (7b)$$

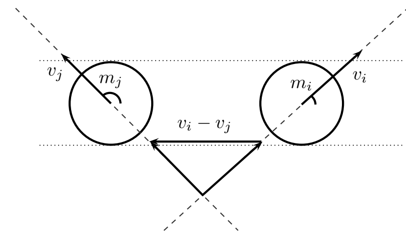


Fig. 3. False conflict case between aircrafts i and j

where $M_1 = -\varepsilon$ is the lower bound of $q_f^+ + q_f^- - \varepsilon$; $M_2 = \overline{v_f} - \underline{v_f}$ is the upper bound of $q_f^+ + q_f^-$ and ε is an infinitesimal parameter. It is easy to see that if there is a velocity change for aircraft $f \in \mathcal{F}$, then $q_f^+ + q_f^- \neq 0$, and the second constraint forces a_f to take the value 1. On the other hand, if there is not a velocity change for aircraft $f \in \mathcal{F}$, then $q_f^+ + q_f^- = 0$, and the first constraint forces a_f to take the value 0.

The next step is changing the value of b_f from 0 to 1 if there is an altitude change. With the following constraint, the number of altitude changes is updated:

$$b_f = 1 - \nu_f^{z_f^*} \quad \forall f \in \mathcal{F},$$

since if aircraft f flies at the same level before and after the current execution, then the variable $\nu_f^{z_f^*}$ will take the value 1 and the difference will be 0, i.e., the aircraft does not change its altitude level. On the other hand, if the aircraft f flies at different altitude levels before and after the current execution, $\nu_f^{z_f^*}$ will be 0 and b_f will take the value 1, i.e., the aircraft changes its altitude level.

False conflicts may happen in the VC model where two aircrafts that are not in conflict are forced by the VC model to change their velocities owing to the fact that the geometric construction permits this situation as can be seen in Fig. 3. In these cases the expression (1) indicates there is a conflict situation but the velocity vectors shows that there is no conflict. To avoid this situation the following parameter is used only when the two straight lines (aircraft trajectories) are secant (in parallel or same straight line either there is no conflict or there is a "head to head" or a "persecution" situation, then fc_{ij} will be one in the first case):

$$fc_{ij} = \begin{cases} 1 & \text{if } d_{ij}^2 - d_{ij}^1 > 0 \wedge d_{ji}^2 - d_{ji}^1 > 0 \wedge sc, hth \neq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $d_{ij}^1 = \text{dist}\{(x_i, y_i), ip_{ij}\}$ and $d_f^2 = \text{dist}\{\big(x_i + \cos(m_i), y_i + \sin(m_i)\big), ip_{ij}\}$ being ip_{ij} the intersection point between the two straight lines considered, that can be calculated during the first step of the algorithm. Notice that with this notation $ip_{ij} = ip_{ji}$.

If $fc_{ij} = 1$ the parameter p_{ij} will be one since if a "false conflict" occurs it is not necessary making a velocity or altitude maneuver that is forced in constraint (20) when the sum of δ variables has to be zero. Notice that fc parameter detects if two aircrafts are getting far among themselves.

Pathological cases may happen in the VC model, since null denominators in expression (1) may appear when implicitly computing the relative velocity vector tangents in the VC model. These pathological cases may cause unstable situations in which conflicts between the involved aircrafts cannot be solved and aircrafts are forced to crash.

One of the most relevant contributions of this paper is the treatment of these pathological cases. The proposed VAC model implicitly detects them in preprocessing. When a pathological case occurs between aircrafts i and j, the conflict is sorted out by turning only the motion angles of the involved aircrafts.

To detect posible pathological cases between two aircrafts in preprocessing a 0-1 parameter is used. This parameter value will be one in case the relative velocity vector tangent tends to infinity and will be zero otherwise. Only pairs of aircrafts with similar abscissa coordinate are considered. The formal definition of the parameter being:

$$pc_{ij} = \begin{cases} 1 & \text{if } |x_i - x_j| \leqslant r_i + r_j \\ 0 & \text{otherwise.} \end{cases}$$

All conflicts where the difference between the abscissas of aircrafts i and j will be taken into account since there might be null denominator. Only in these cases, considering aircrafts i and j, parameters m_i^* and m_j^* must be taken as $m_i^* = m_i^* + \frac{\pi}{2}$ and $m_j^* = m_j^* + \frac{\pi}{2}$, i.e., the elements of the configurations are turned $\frac{\pi}{2}$ radians since after this turn $|x_i - x_j| > 0$ if before existed a pathological case. The idea is turning implicitly these two configurations but only in the constraints where aircrafts i and j are considered. This does not mean that aircrafts make a heading angle change, it means that the two aircrafts involved in a pathological case are compared in the constraint set as a turn of all elements has been made. It has to be distinguished what angle (the initial or the turned angle) has to be considered by using the pc parameter. The constraints in each case of the VC model are rewritten as follows:

$$\begin{aligned}(v_i + q_i)\big(\cos(m_i^*)(1 - pc_{ij}) - \sin(m_i^*)pc_{ij}\big) - \\ (v_j + q_j)\big(\cos(m_j^*)(1 - pc_{ij}) - \sin(m_j^*)pc_{ij}\big) \\ \leqslant M'_m(1 - \delta_{ijz}^n)\end{aligned}$$

$$\begin{aligned}-(v_i + q_i)\big(h_i(1 - pc_{ij}) + \hat{h}_i pc_{ij}\big) + \\ (v_j + q_j)\big(h_j(1 - pc_{ij}) + \hat{h}_j pc_{ij}\big) \\ \leqslant M''_m(1 - \delta_{ijz}^n),\end{aligned}$$

where $\cos(m_i^* + \frac{\pi}{2}) = -\sin(m_i^*)$, M'_m and M''_m are the upper bounds of the constraints; $n = 1, \ldots, 4$; $M'_m = (\overline{v_i} + \overline{v_j})$ and $\hat{h}_i$ and $\hat{h}_j$ are the new parameters h_i and h_j built by using the new turned angles. The M''_m value in these constraints will be as follows:

$$M''_m = (\overline{v_i}|h_i| + \overline{v_j}|h_j|)(1 - pc_{ij}) + (\overline{v_i}|\hat{h}_i| + \overline{v_j}|\hat{h}_j|)pc_{ij}.$$

Finally, the objective function will be constructed including all factors and different objectives that are relevant for the problem. They are detailed below.

First of all, the first objective consists of minimizing the velocity changes absolute value, such that velocity changes and an early arrival or an arrival delay to destination point in each aerial aircraft sector are smoothed. In [8] this term was considered by maximizing the velocity variations, which does not make sense when aircrafts fly with an intermediate velocity and they have to be accelerated due to this objective function. To make early arrivals or arrival delays as small as possible, the minimization of the velocity variations using the absolute value function to avoid high changes in the initial flight plan is proposed. This is done as follows,

$$\min \sum_{f \in \mathcal{F}} |q_f| = \min \sum_{f \in \mathcal{F}} (c_f^{q^+} q_f^+ + c_f^{q^-} q_f^-).$$

Different objectives are considered but with different magnitudes. All magnitudes involved in velocity changes expressions are normalized (between 0 and 1) as follows,

$$\min \sum_{f \in \mathcal{F}} \Big(\frac{c_f^{q^+} q_f^+}{\overline{v}_f - \underline{v}_f} + \frac{c_f^{q^-} q_f^-}{\overline{v}_f - \underline{v}_f} \Big).$$

Only q_f^+ or q_f^- will take value greater than zero, and the objective function will make one of these variables equal to zero. Notice that $\overline{v}_f - \underline{v}_f$ is the upper bound of a velocity variation for an aircraft $f \in \mathcal{F}$.

In case of a very congested airspace, an aircraft may need to climb or descend more than one level thus provoking unfeasible situations. A new term is introduced in the objective function to avoid unfeasibility. This option is modeled by adding the following term to the objective function as done with (6), such that constraint (4) will be removed,

$$\min \sum_{f \in \mathcal{F}} c_f^j \rho_f.$$

Another term to minimize is the number of velocity and altitude changes done by each aircraft in the aerial sector. Notice that the costs will increase when several changes are done in the same aircraft. The function is as follows:

$$\min \sum_{f \in \mathcal{F}} (c_f^v n_f^v a_f + c_f^a n_f^a b_f).$$

Returning all aircraft configurations to the initial ones, both in velocity and altitude is desirable. A new term can be included so that the aircrafts return again to their initial configurations when they are not in conflict. To model this term, the difference between the current velocity and the initial velocity configuration in the initial flight plan is penalized. A new term can also be included to penalize the difference between the current level and the initial level configuration with the ν variable.

The objective relative to the return to the initial velocity configuration consists in forcing the aircrafts to arrive at the destination sector point in the predicted time. For this purpose, some observations about geometric constructions are necessary; take Fig. 4 as support.

The distance between points (x_f, y_f) and (x_f^1, y_f^1) can be known in two ways:

$$dist((x_f, y_f), (x_f^1, y_f^1)) = \hat{v}_f(t_1 - t)$$
$$dist((x_f, y_f), (x_f^1, y_f^1)) = \sqrt{(x_f^1 - x_f)^2 + (y_f^1 - y_f)^2}.$$

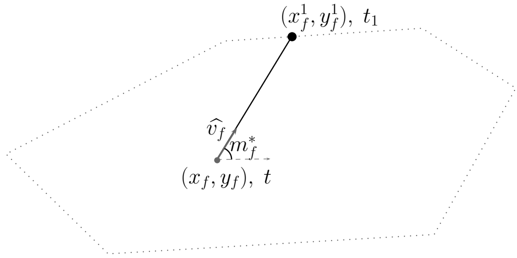


Fig. 4. Return to the initial configuration time

Hence,

$$\hat{v}_f(t_1 - t) = \sqrt{(x_f^1 - x_f)^2 + (y_f^1 - y_f)^2}.$$

where the velocity an aircraft might have to arrive at the destination point in the predicted time is:

$$\widehat{v_f} = \frac{\sqrt{(x_f^1 - x_f)^2 + (y_f^1 - y_f)^2}}{t_1 - t}.$$

Therefore, the difference between current and optimal velocities to arrive at the destination point in the predicted time is penalized as follows:

$$\min \sum_{f \in \mathcal{F}} c_f^{\hat{v}} |v_f + q_f - \widehat{v_f}| = \min \sum_{f \in \mathcal{F}} c_f^{\hat{v}} \beta_f.$$

This function is non linear, but it can be modeled as a linear function by using a maximum function as in (6). The addend can be descomposed as $\beta_f = v_f + q_f - \widehat{v_f} = (v_f + q_f^+) - (q_f^- + \widehat{v_f})$, and using it, two additional constraints are necessary:

$$\beta_f \geqslant (v_f + q_f^+) - (q_f^- + \widehat{v_f})$$
$$\beta_f \geqslant (q_f^- + \widehat{v_f}) - (v_f + q_f^+).$$

A term to penalize the difference between the current level and the initial level configuration is added with the ν variable. An aircraft takes its initial level configuration if $\nu_f^{z_f^*} = 1$, therefore, the new objective function will be as follows,

$$\max \sum_{f \in \mathcal{F}} c_f^{z^*} \nu_f^{z_f^*} = \min \sum_{f \in \mathcal{F}} -c_f^{z^*} \nu_f^{z_f^*}.$$

Moreover, the objective function terms and some new costs to model other preferences like fuel consumption can be added. Also, any linear combination of all them is possible just as considering weights w_n for each objective function term $n = 1, \ldots, 5$ to attribute the importance given to each term.

V. Problem Formulation

Next, the full VAC model is presented, where all unstable situations are considered and solved.

$$\min w_1 \sum_{f \in \mathcal{F}} \left(\frac{c_f^{q^+} q_f^+}{\overline{v}_f - \underline{v}_f} + \frac{c_f^{q^-} q_f^-}{\overline{v}_f - \underline{v}_f} \right) + w_2 \sum_{f \in \mathcal{F}} c_f^j \rho_f + w_3 \sum_{f \in \mathcal{F}} (c_f^v n_f^v a_f + c_f^a n_f^a b_f) + w_4 \sum_{f \in \mathcal{F}} c_f^{\hat{v}} \beta_f - w_5 \sum_{f \in \mathcal{F}} c_f^{z^*} \nu_f^{z_f^*} \quad (10)$$

subject to:

$$\underline{v_f} \leqslant v_f + q_f \leqslant \overline{v_f} \quad \forall f \in \mathcal{F} \quad (11)$$

$\forall i, j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j$:

$$(v_i + q_i)\big(\cos(m_i^*)(1 - pc_{ij}) - \sin(m_i^*) pc_{ij}\big) - (v_j + q_j)\big(\cos(m_j^*)(1 - pc_{ij}) - \sin(m_j^*) pc_{ij}\big) \leqslant (\overline{v_i} + \overline{v_j})(1 - \delta_{ijz}^1) \quad (12)$$

$$-(v_i + q_i)\big(h_i(1 - pc_{ij}) + \hat{h}_i pc_{ij}\big) + (v_j + q_j)\big(h_j(1 - pc_{ij}) + \hat{h}_j pc_{ij}\big) \leqslant \big((\overline{v_i}|h_i| + \overline{v_j}|h_j|)(1 - pc_{ij}) + (\overline{v_i}|\hat{h}_i| + \overline{v_j}|\hat{h}_j|) pc_{ij}\big)(1 - \delta_{ijz}^1) \quad (13)$$

$$(v_i + q_i)\big(\cos(m_i^*)(1 - pc_{ij}) - \sin(m_i^*) pc_{ij}\big) - (v_j + q_j)\big(\cos(m_j^*)(1 - pc_{ij}) - \sin(m_j^*) pc_{ij}\big) \leqslant (\overline{v_i} + \overline{v_j})(1 - \delta_{ijz}^2) \quad (14)$$

$$(v_i + q_i)\big(k_i(1 - pc_{ij}) + \hat{k}_i pc_{ij}\big) - (v_j + q_j)\big(k_j(1 - pc_{ij}) + \hat{k}_j pc_{ij}\big) \leqslant \big((\overline{v_i}|k_i| + \overline{v_j}|k_j|)(1 - pc_{ij}) + (\overline{v_i}|\hat{k}_i| + \overline{v_j}|\hat{k}_j|) pc_{ij}\big)(1 - \delta_{ijz}^2) \quad (15)$$

$$-(v_i + q_i)\big(\cos(m_i^*)(1 - pc_{ij}) - \sin(m_i^*) pc_{ij}\big) + (v_j + q_j)\big(\cos(m_j^*)(1 - pc_{ij}) - \sin(m_j^*) pc_{ij}\big) \leqslant (\overline{v_i} + \overline{v_j})(1 - \delta_{ijz}^3) \quad (16)$$

$$(v_i + q_i)\big(h_i(1 - pc_{ij}) + \hat{h}_i pc_{ij}\big) - (v_j + q_j)\big(h_j(1 - pc_{ij} + \hat{h}_j pc_{ij})\big) \leqslant \big((\overline{v_i}|h_i| + \overline{v_j}|h_j|)(1 - pc_{ij}) + (\overline{v_i}|\hat{h}_i| + \overline{v_j}|\hat{h}_j|) pc_{ij}\big)(1 - \delta_{ijz}^3) \quad (17)$$

$$-(v_i + q_i)\big(\cos(m_i^*)(1 - pc_{ij}) - \sin(m_i^*) pc_{ij}\big) + (v_j + q_j)\big(\cos(m_j^*)(1 - pc_{ij}) - \sin(m_j^*) pc_{ij}\big) \leqslant (\overline{v_i} + \overline{v_j})(1 - \delta_{ijz}^4) \quad (18)$$

$$-(v_i + q_i)\big(k_i(1 - pc_{ij}) + \hat{k}_i pc_{ij}\big) + (v_j + q_j)\big(k_j(1 - pc_{ij}) + \hat{k}_j pc_{ij}\big) \leqslant \big((\overline{v_i}|k_i| + \overline{v_j}|k_j|)(1 - pc_{ij}) + (\overline{v_i}|\hat{k}_i| + \overline{v_j}|\hat{k}_j|) pc_{ij}\big)(1 - \delta_{ijz}^4) \quad (19)$$

$$\delta_{ijz}^1 + \delta_{ijz}^2 + \delta_{ijz}^3 + \delta_{ijz}^4 + \delta_{ijz}^5 = 1 - p_{ij} \quad (20)$$

$\forall i, j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}^i \cap \mathcal{Z}^j$:

$$\delta_{ijz}^5 = 1 \qquad \text{if } hth_{ij} + sc_{ij} \geqslant 1 \quad (21)$$

$$\nu_i^z + \nu_j^z \geqslant \delta_{ijz}^5 - 1 \quad (22)$$

$$\nu_i^z + \nu_j^z \leqslant 2 - \delta_{ijz}^5 \quad (23)$$

TABLE I
VAC MODEL DIMENSION

Variables q_f^+: F	Variables q_f^-: F
Variables ν_f^z: FZ	Variables a_f: F
Variables b_f: F	Variables ρ_f: F
Variables β_f: F	Variables δ_{ijz}^n: $5Z\frac{(F-1)F}{2}$

(a) Number of variables

C. (11): $2F$	C. (12)-(20): $9Z\frac{F(F-1)}{2}$
C. (21): $Z\frac{F(F-1)}{2}$	C. (22)-(23): $ZF(F-1)$
C. (24): $2F$	C. (25)-(26): $2F$
C. (27): F	C. (28)-(29): $2F$
C. (30)-(31): $2F$	

(b) Number of constraints

$\forall f \in \mathcal{F}$:

$$\sum_{z \in \mathcal{Z}^f} \nu_f^z = 1 \tag{24}$$

$$-\varepsilon(1-a_f) + \varepsilon \leqslant q_f^+ + q_f^- \tag{25}$$

$$q_f^+ + q_f^- \leqslant (\overline{v_f} - \underline{v_f})a_f \tag{26}$$

$$1 - \nu_f^{z_f} = b_f \tag{27}$$

$$\sum_{z \in \mathcal{Z}^f} z\nu_f^z - z_f \leqslant \rho_f \tag{28}$$

$$z_f - \sum_{z \in \mathcal{Z}^f} z\nu_f^z \leqslant \rho_f \tag{29}$$

$$(v_f + q_f^+) - (q_f^- + \widehat{v_f}) \leqslant \beta_f \tag{30}$$

$$(q_f^- + \widehat{v_f}) - (v_f + q_f^+) \leqslant \beta_f \tag{31}$$

$\forall f \in \mathcal{F}$:

$$q_f \in \mathbb{R} \tag{32}$$

$$q_f^+, q_f^-, \beta_f \in \mathbb{R}^+ \tag{33}$$

$$\rho_f \in \mathbb{Z}^+ \tag{34}$$

$\forall f, i, j \in \mathcal{F} : i < j, \forall z \in \mathcal{Z}$:

$$\nu_f^z, a_f, b_f, \delta_{ijz}^1, \delta_{ijz}^2, \delta_{ijz}^3, \delta_{ijz}^4, \delta_{ijz}^5 \in \{0,1\} \tag{35}$$

The objective function (10) contemplates the different terms explained above; constraint (11) contains the velocity bounds; constraints (12)-(20) detect conflicts that can be solved with velocity changes; constraints (21)-(23) include altitude changes in the model; constraint (24) forces aircrafts to fly at one and only one altitude level; constraints (25)-(27) update the number of changes in velocity and altitude respectively; constraints (28)-(29) are for the number of altitude levels that an aircraft ascends or descends; constraints (30)-(31) force aircrafts to exit from the aerial sector at the predicted time; expressions (32)-(35) represent the type of variables in the model.

Table I shows the dimension of the VAC model. Notice that model dimension depends on the objective function of choice.

VI. Case Studies

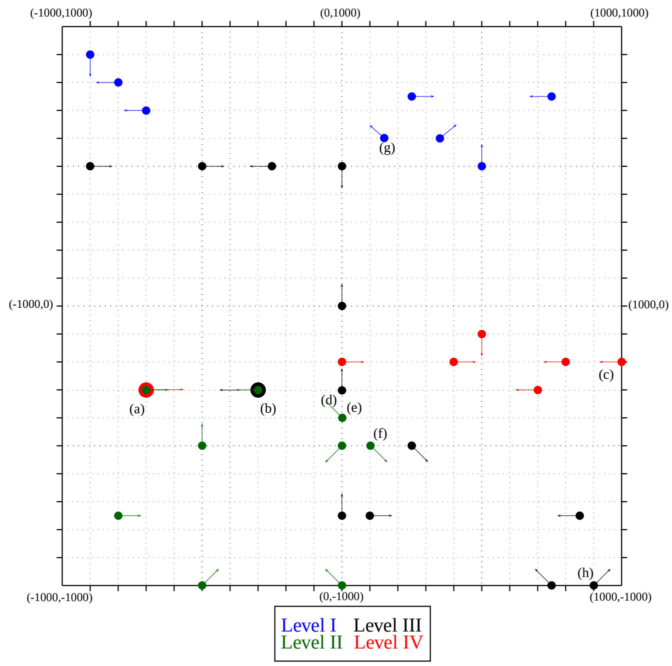
The VAC model has been initially tested with the case shown in Fig. 5(a) by using weights $w_1 = w_2 = 0.5$ and $w_3 = w_4 = w_5 = 0$, where velocity variations and the levels that the aircrafts have to change are minimized. In this objective function $c_f^{q^+} = 1$, $c_f^{q^-} = 1$ and $c_f^j = 1$, $\forall f \in \mathcal{F}$. The parameter p_{ij} is not considered unless the "false conflict" situation occurs, therefore all pairs of aircrafts are taken into account. Notice that we are considering $F = 37$ aircrafts in possible conflict. Note: There are two aircrafts in each point (a) and (b); All aircrafts fly with the same velocity, except aircrafts (c) and (d) that fly with a higher velocity to test "pursuit cases"; Aircrafts (e), (f), (g) and (h) are in "false conflict" that implies an unnecessary velocity change.

Fig. 5(b) depicts the results of applying the VAC model for collision avoidance in the case depicted in Fig. 5(a) as follows:

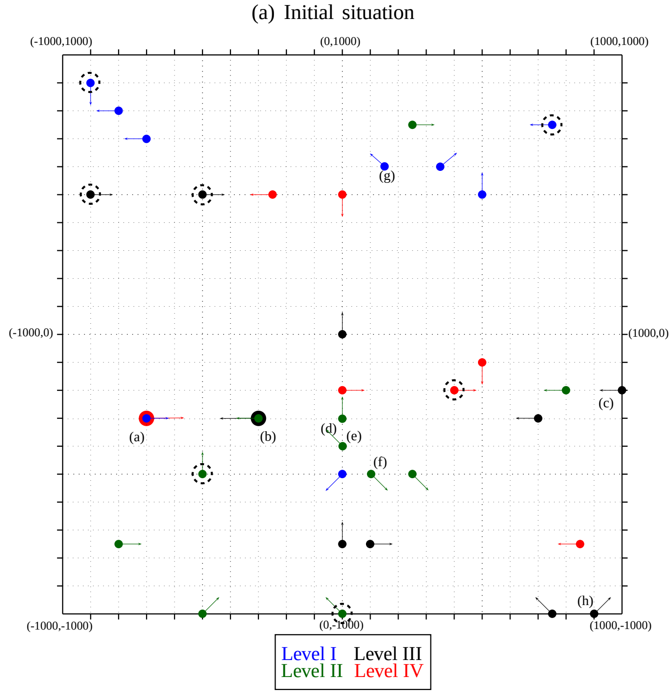
- 7 velocity changes have been performed. The dotted line circles in the figure denote the positive variation of velocity, and the dashed line circle denotes the negative variation of velocity. In this case, all velocity changes are negative (slowdown).
- The points (a) and (b) still have two aircrafts each, but the conflict has been avoided.
- The aircrafts (e), (f), (g) and (h) have not changed their velocities since they are in a "false conflict".
- There are 11 altitude level changes, 4 positive changes and 7 negative changes, one of which descends two levels.
- The number of constraints, and continuous and 0-1 variables in the reduced MIP have been 2998, 37 and 1963, respectively.
- The objective function value is 6.5562.
- The execution time has been 8.42 seconds by using the optimization engine CPLEX v.12.1 [12] (with the default options) in the following HW/SW platform: Intel Core 2DUO P8400, 2.26GHz, 2GB RAM; Microsoft Windows XP Professional SO.

Next, some computational experience for the VAC model is reported. 25 random simulations are performed for each dimensional case, and the result averages are presented. Table II shows the dimensions of the model, whereas Table III reports the most important results. The headings are as follows: *Case*: Gives the case configuration: CAAA-ZZ denotes number of aircrafts (AAA) and levels (ZZ); m: Number of constraints; n: Number of variables; d: Constraint matrix density; m^*, n^* and d^*: the number of constraints, variables and constraint matrix density, respectively, after CPLEX preprocessing; z_{lp}: Value of the objective function in the continuous linear relaxation; z_s: Value of the bound after performing the CPLEX cut identification and appending at node 0; z_{ip}: Value of the objective function for the optimal solution of the problem; GAP_{lp}: $\frac{z_{ip}-z_{lp}}{z_{ip}}$ %; GAP_s: $\frac{z_{ip}-z_s}{z_{ip}}$ %; nb: Number of times that there is branching; nn: Number of CPLEX branch-and-cut nodes; t_{lp}: Time (secs.) to obtain the z_{lp} value; t_s: Time (secs.) to obtain the z_s value; t_{ip}: Time (secs.) to obtain the z_{ip} value; nc: Total number of cuts performed by CPLEX. Note: The minimum, average and maximum GAPs are reported in Table III.

In a realistic case, the number of aircrafts and levels vary between 15 and 30 and between 1 and 10 levels, respectively. The airspace under consideration has 50x50 squared units



(a) Initial situation



(b) Results for the VAC model in case of Fig. 5(a)

Fig. 5. Testing the VAC model.

leading to a greater number of conflicts while the dimensions are increased. The weights under consideration are $w_1 = \cdots = w_5 = 0.2$.

The first computational observation that can be made in Table II is the strength of the CPLEX preprocessing by comparing the columns m and m^* and n and n^*. Moreover, the dimensions m^* and n^* are still very big. In Table III the very small GAP_s for all the instances can be observed, showing the tightness of the model plus the CPLEX cuts. (Notice that nb is small for most of the instances) and, then, the elapsed time is very small.

TABLE II
DIMENSIONS TABLE.

Case	m	n	d	m^*	n^*	d^*
C020-05	10227.0	4773.1	0.0010	1312.0	803.4	0.0056
C020-07	14830.0	6910.0	0.0007	1838.8	1121.3	0.0041
C020-10	21100.0	9820.0	0.0005	2643.1	1587.1	0.0029
C025-05	16750.0	7775.0	0.0006	1996.8	1205.4	0.0037
C025-07	23350.0	10825.0	0.0004	2866.4	1683.6	0.0027
C025-10	33250.0	15400.0	0.0003	4207.2	2448.6	0.0019
C030-05	24225.0	11205.0	0.0004	3054.0	1777.4	0.0025
C030-07	33795.0	15615.0	0.0003	3931.2	2292.7	0.0020
C030-10	48150.0	22230.0	0.0002	5403.9	3157.4	0.0015
C035-05	33075.0	15260.0	0.0003	3960.6	2290.1	0.0020
C035-07	46165.0	21280.0	0.0002	5411.5	3119.7	0.0015
C035-10	65800.0	30310.0	0.0002	7206.3	4132.8	0.0011
C040-05	43300.0	19940.0	0.0002	5453.7	3088.5	0.0015
C040-07	60460.0	27820.0	0.0002	6812.5	3919.8	0.0012
C040-10	86200.0	39640.0	0.0001	9981.9	5677.5	0.0008
C045-05	54900.0	25245.0	0.0002	6157.8	3505.6	0.0013
C045-07	76680.0	35235.0	0.0001	9035.1	5088.6	0.0009
C045-10	109350.0	50220.0	0.0001	12474.6	6998.8	0.0007
C050-05	67875.0	31175.0	0.0001	7663.0	4326.1	0.0010
C050-07	94825.0	43525.0	0.0001	10831.9	6077.8	0.0008
C050-10	135250.0	62050.0	0.0001	16204.9	9009.2	0.0005

VII. CONCLUSION

The so-called VAC model, for the resolution of the Collision Avoidance Problem has been presented. It adds to the VC model [8] new interesting features, the most important being the inclusion of altitude changes to avoid infeasible improves brought by velocity bounds. Also, the VAC model completes the VC model in the sense that it takes into account null denominator appearances and "false conflict" situations just as different safety radii consideration. This model looks for an equilibrium in the number of maneuvers for each aircraft, penalizing those ones with many maneuvers to realize. The elapsed times are very small and, then, the model could be applied in real time, helping ATC decision making.

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Antonio Alonso Ayuso was born in Santander, Spain, in 1968. He received the Msc. in Mathematics in 1992 and the Ph.D. degree in Mathematics in 1997, in University Complutense, Madrid, Spain. He is currently full time professor at the Dep. of Statistics and Operational Research at Rey Juan Carlos University of Madrid. He has been member of several research projects at different Spanish Universities, European Commission (Leonardo Da Vinci program) and National Research Plan (several of them as main research). He has a number of papers in well rated international journal and has collaborate with different firms in Applied Projects.

His main research interests include linear and integer mathematical programming, decision models and stochastic programming applied to combinatorial problems.

Laureano F. Escudero was born in Valladolid, Spain, in 1942. PhD. degree in Economics, Universidad de Deusto, Bilbao, Spain, in 1974,. Currently, he is Professor of Statistics and Operations Research at the University Rey Juan Carlos, Spain. In the period 2003-04 he was the President of EURO (Association of European Operations Research Societies). He has worked at IBM Research, Scientific and Development Centers in Madrid (Spain), Palo Alto (California), Sindelfingen (Germany) and Yorktown Heights (NY), 1972-1991. He taught Mathematical Programming at the Mathematical Sciences School, University Complutense of Madrid, 1992-2000 and Stochastic Programming at the University Miguel Hernández, Spain. He is the author of several books and more than 100 scientific papers published in leading journals.

His main research interest includes different mathematical programming fields (linear, integer, nonlinear, stochastic) and their applications.

Francisco Javier Martín Campo was born in Salamanca, Spain, in 1983. Msc. in Mathematics, University Complutense, Madrid, Spain in 2007. He is currently pursuing the Ph.D. degree in mathematics at Rey Juan Carlos University. At present he is also working in a project based on air traffic flow management and collision avoidance so-called Atlantida from a contract with the company GMV Aerospace and Defence S.A., on the same subject of the thesis as well as in the PLANIN project (Planning under uncertainty).

His main research interests include optimization, mixed integer linear programming models and algorithms, and air traffic management models.

TABLE III
RESULTS TABLE.

Case	z_{lp}	z_s	z_{ip}	GAP_{lp}	GAP_s	nb	nn	t_{lp}	t_s	t_{ip}	nc
C020-05	5.8940	6.1448	6.1548	0.00 4.24 100.00	0.00 0.16 1.74	0	0.00	0.03	0.09	0.09	10.9
C020-07	5.1651	5.4854	5.5090	0.00 6.24 24.54	0.00 0.43 11.62	2	15.00	0.04	0.14	0.14	33.9
C020-10	4.1421	4.2713	4.2951	0.00 3.56 12.41	0.00 0.55 4.35	4	14.75	0.06	0.19	0.19	18.9
C025-05	9.7815	10.5492	10.6043	0.00 7.76 52.36	0.00 0.52 3.51	3	65.00	0.05	0.18	0.20	52.7
C025-07	5.2982	5.4847	5.4943	0.00 3.57 41.63	0.00 0.17 1.50	1	20.00	0.06	0.22	0.23	37.1
C025-10	7.7335	8.1448	8.1726	0.00 5.37 42.96	0.00 0.34 2.06	1	20.00	0.10	0.28	0.29	45.3
C030-05	9.3827	10.6305	10.6904	0.00 12.23 100.00	0.00 0.56 67.79	6	22.33	0.07	0.35	0.38	103.2
C030-07	6.2307	6.9476	6.9917	0.00 10.88 100.00	0.00 0.63 6.56	3	11.67	0.10	0.36	0.38	63.0
C030-10	5.0408	5.3576	5.3877	0.00 6.44 79.61	0.00 0.56 60.63	4	11.50	0.14	0.44	0.45	53.6
C035-05	12.1933	13.6593	13.7500	0.00 11.32 57.01	0.00 0.66 7.19	11	15.55	0.10	0.52	0.59	188.1
C035-07	10.8919	12.2389	12.3179	0.00 11.58 100.00	0.00 0.64 8.95	3	12.00	0.14	0.50	0.54	139.3
C035-10	9.8494	10.5824	10.6036	0.00 7.11 41.72	0.00 0.20 9.42	2	2.50	0.21	0.61	0.61	132.4
C040-05	13.3918	15.8004	15.9059	0.00 15.81 76.51	0.00 0.66 3.87	9	21.67	0.14	0.67	0.84	229.4
C040-07	12.1882	14.0154	14.1239	0.00 13.71 85.95	0.00 0.77 5.14	7	33.71	0.19	0.84	1.01	263.6
C040-10	8.0451	8.9149	8.9472	0.00 10.08 47.74	0.00 0.36 6.26	8	21.63	0.28	0.96	1.12	219.2
C045-05	9.5783	12.4974	12.6698	0.00 24.40 100.00	0.00 1.36 7.45	12	55.33	0.18	0.88	1.06	320.6
C045-07	7.3346	9.2808	9.4741	0.00 22.58 100.00	0.00 2.04 13.55	11	32.45	0.25	1.20	1.52	284.4
C045-10	11.7084	12.9528	12.9969	0.00 9.91 93.13	0.00 0.34 3.06	4	18.25	0.37	1.15	1.23	241.1
C050-05	12.5373	15.7573	15.9703	0.00 21.50 100.00	0.00 1.33 75.43	13	68.08	0.23	1.29	1.65	325.3
C050-07	12.4788	14.9225	15.0525	0.00 17.10 100.00	0.00 0.86 5.21	6	43.50	0.32	1.25	1.63	377.2
C050-10	9.1112	10.9886	11.0690	0.00 17.69 100.00	0.00 0.73 30.51	6	92.33	0.47	1.78	2.32	415.5