Dynamic Pricing of Perishable Assets under Competition

Guillermo Gallego* • Ming Hu†

Department of Industrial Engineering and Operations Research,

Columbia University, New York, NY 10027, USA

gmg2@columbia.edu • *mh2252@columbia.edu*

Abstract

Legacy carriers in the airline industry are facing fierce competition from low cost providers that impose few or no fare restrictions. In addition, the Internet has provided customers with instant fare transparency. As a result, existing revenue management solutions based on allocating capacity to different fare classes are becoming less effective. To avoid revenue erosion from inadequate solutions, carriers need to develop new systems based on how consumers behave given the information available to them. This requires a choice-based, multi-player, game theoretic formulation of dynamically pricing perishable inventories over finite-horizons. Here we present such a formulation as a stochastic control problem in continuous time and provide sufficient conditions for the existence and uniqueness of open-loop Nash equilibria of the corresponding differential game. We show that pricing heuristics suggested by the open-loop Nash equilibria are asymptotically ϵ-Nash equilibria for the stochastic game. The robust stability of the unique open-loop Nash equilibrium and repeated versions of the single-stage pricing game are also studied. Asymptotic results are extended to network models, to sales over different channels and to account for quality attributes.

Key words: oligopoly pricing; dynamic pricing; perishable asset; finite horizon; Nash equilibrium; supermodular game; revenue management

Subject classifications: Transportation: yield management, pricing. Dynamic programming/optimal

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1 Introduction

1.1 Motivation

Current revenue management (RM) practice, in spite of its success and popularity, is based on assumptions that no longer hold in practice. The most critical flaw is that traditional revenue management (or yield management) models are not designed around the pricing and quality attributes of the products available to consumers at the time of purchase. Legacy RM systems are designed under the assumption that demands for different fare classes are independent random variables with a pricing team designing fares and another team allocating capacity to fare classes. This flaw is exacerbated by low cost carriers offering low fares with few or no restrictions and by the price transparency enabled by the Internet.

While carriers and solution providers agree that new solutions are needed to stem revenue erosion, there is no agreement as to what needs to be done. Some solution providers have proposed an incremental approach that keeps pricing and capacity allocation separate acknowledging that most of the demand will go to the lowest open fare. Aggregate sales forecasts, however, ignore detailed pricing by competitors. At the other end of the spectrum people are trying to integrate pricing and capacity allocation into a single system that takes into account pricing and quality attributes of the set of products available to customers at the time of purchase. This paper studies how such a system will behave by analyzing a stochastic dynamic pricing model of perishable assets under competition. The results presented here should be of interest to both RM researchers and practitioners.

The last few years have witnessed an evolution of demand modeling in RM. Standard quantity-based RM models (see Talluri and van Ryzin (2004b, Chap.2,3)) assume that demand for each product is an independent stochastic process. These independent-demand models consider neither customer behavior nor competition, but worked well before fare transparency and low-cost carriers. The current market environment makes the independent demand assumption untenable as explained in Cooper et al. (2006). Demand choice models that ignore competition, see Talluri and van Ryzin (2004a) for the single leg case and Gallego et al. (2004) and van Ryzin and Liu (2004) for the network case, are a step closer to reality as they take customer choice behavior into account. In this paper, we consider a demand model that captures consumer choice in a competitive environment. See also Shankar et al. (2005).

Quantity-based RM practice separates pricing and capacity allocation decisions. Fares are designed by the pricing team and capacity is allocated to the different fares depending on how demand unfolds. Most of the literature in RM deals with the issue of capacity allocation. Pricing decisions are kept at a more *strategic* level but to our knowledge there is little science behind current pricing practices. This separation limits the ability to price more effectively by dynamically reacting to changes in prices offered by competitors. We believe that appropriately applied dynamic pricing can "ration" capacity more profitably than by limiting supply in a fare-based RM system. This is due to the following two facts. First, dynamic pricing takes into account competitors' prices in real time resulting in more accurate sales forecasts and second, dynamic pricing is not limited to a pre-specified menu of fares. By having more liberty to price a product and by basing prices on current competitors' prices, dynamic pricing can be more effective than capacity allocation with a given fare structure.

Historically, there was a reason to separate pricing and capacity allocation decisions. Indeed, partially due to difficulties collecting competitors' prices and to the complexity of analyzing competitive models, most legacy RM models were confined to monopolistic settings. However, recent development of search engine technology makes competitors' prices instantaneously available to consumers as well as ready to be fed into competitive pricing models as input data. Indeed, online travel sites such as Expedia, Hotwire, Orbitz, Priceline and Travelocity gather information and list flight, hotel, car rental and cruise fares in real time among competitors across the travel industries. Price search engines such as Froogle.com provide real-time product prices both online and in local stores.

Let us consider a concrete example to motivate our study. On June 15, 2006, a potential traveler used a search engine for a one-way airline ticket of Economy/Coach class from New York City, NY (LGA) to Los Angeles, CA (LAX) for on June 20, 2006 and 6:00 AM target departure time. The search results are summarized in Table 1. We make the following observations and insights through this example. First, price transparency offered by travel sites facilitates customers' comparison shopping and requires a demand choice model under competition to more fully account for the choices available. Second, it is easy for airlines to monitor competitors' real-time pricing and even capacity levels[1]. Third, we can consider similar flights offered by different airlines as the *same* product in our choice model by embedding attributes such as departure time, number of connections, flight duration and presence of origin into the demand choice model. Finally, current pricing practice lags behind academic research since it ignores crucial differentiated attributes in the pricing. In this example, United, AA, Continental and Delta offered the same price although their flight durations are different. It would seems possible for United Airlines to charge a higher price given it offers the shortest flight.

Airlines	Flight	Connection	Departure	Duration	Price
United	115/247	Denver (Denver Intl.)	6:00 am	7hr 06mn	\$389
American Airlines	301/549	Chicago (ORD)	6:03 am	7hr 31mn	\$389
Continental	633/1495	Houston (IAH)	5:50 am	7hr 52mn	\$389
Delta	935/907	Atlanta (Hartsfield Intl.)	6:00 am	8hr 10mn	\$389
ATA	4219/259	Chicago (MDW)	6:00 am	8hr 40mn	\$308
US Airways	31	N/A (1 stop)	6:40 am	7hr 58mn	\$382

Table 1: Expedia.com Prices on 06/15/06 of a 06/20/06 One-Way Economy Ticket LGA to LAX

In this paper we assume that multiple capacity providers compete to sell their own fixed initial inventories of perishable items over a finite sales horizon. We assume there are no replenishment opportunities. Demand is modeled as a non-homogeneous Poisson process that depends on the current time and the prices offered by all players[2]. We will address the following issues concerning this non-cooperative dynamic pricing game. How should each player dynamically price its inventories under the competition? Is there a tractable heuristic for each player that has good asymptotic properties? If there exists such a heuristic, under what conditions can we guarantee its existence and uniqueness? What if some players follow a strategy that is different from the equilibrium strategy? How does this affect them and other players? What if the game is played again and again?

[1] "Preview seat availability" function is available for most flights.

[2] In the basic model, we assume the instantaneous demand rate $\lambda(t)$ is a homogeneous function of the price vector across the industry, i.e., $\lambda(t) = \lambda\big(p(t)\big)$, however, the price vector could be time-dependent. We will extend the demand rate function to a non-homogeneous function in §6.2.

1.2 Literature Review

Dynamic pricing of perishable assets over a finite horizon for a monopolist has been intensively and extensively studied in the literature. In general, stochastic dynamic pricing problems are hard to solve and only structural results and asymptotic optimal heuristics can be obtained. Kincaid and Darling (1963) initialize the study by analyzing a continuous-time dynamic pricing problem of single-commodity with a finite deadline and provide sufficient conditions for the optimal strategies. Gallego and van Ryzin (1994) formulate the dynamic pricing problem as an intensity control problem and obtain structural monotonicity results for the optimal Markovian policy. More importantly, this paper proves the asymptotic optimality of the fixed-price heuristic suggested by the corresponding deterministic problem and bounds the additional expected revenue one can obtain from a dynamic pricing policy over the fixed-price policy. Gallego and van Ryzin (1997) extend the single-product pricing problem to a multi-product dynamic pricing problem and give its applications to network yield management. Bitran and Mondschein (1997) consider a discrete-time dynamic pricing model to incorporate periodic pricing reviews and monotonically decreasing price path constraints. Zhao and Zheng (2000) propose a continuous-time dynamic pricing model with non-homogeneous demand and identify a sufficient condition to guarantee monotonically decreasing prices. However, in order to solve stochastic dynamic pricing problems, more specific assumptions have to be made. Feng and Gallego (1995) obtain a threshold-of-time optimal Markovian pricing policy if only a single price change from a given initial price to another given price is allowed. Feng and Gallego (2000) characterize the optimal timing of price changes within a given menu of allowable price paths. Feng and Xiao (2000a,b) identify the optimal pricing policy in a close form under the assumption that the allowable price set is discrete and finite, and the price changes are either reversible or irreversible.

There is a growing body of literature on RM under competition. Netessine and Shumsky (2005) examine competitive quantity-based RM models, both single-leg and multi-leg models, and provide a general sufficient condition under which a pure-strategy Nash equilibrium exists in these RM games. Gallego et al. (2005b) study both sequential game and infinitely repeated simultaneous game between two competitors that sell fixed capacities in both a forward and a spot market.

Oligopoly pricing, common in the economics and marketing literature, is gaining currency within the RM community. One stream of research characterizes the market equilibrium by the methodology of *quasi-variational inequalities (QVI)* and shows existence and uniqueness of the equilibrium policies under conditions adapted from the *QVI* literature. This line of research considers combined price-based and quantity-based RM problems. Perakis and Sood (2006) study a multi-period competitive dynamic pricing model of a single perishable asset with fixed inventory and address demand uncertainty using ideas from *robust optimization*. Nguyen and Perakis (2005) extend the single-asset model to a multi-product competitive pricing game and present an iterative learning algorithm to compute the market equilibrium policies. Mookherjee and Friesz (2005) consider a combined pricing, resource allocation, overbooking RM problem over networks as well as under competition. The other stream falls under the framework of *supermodular games (SG)* and provides sufficient conditions to guarantee existence and uniqueness of the equilibrium that are equivalent to supermodularity and "diagonal dominance" conditions respectively in terms of *SG*. Following this line, Bernstein and Federgruen produced a series of papers considering joint pricing and inventory decisions for competing retailers from a *supply chain* perspective. Bernstein and Federgruen (2003) consider a two-echelon distribution system with a supplier and multiple competing retailers. The pricing decisions and inventory strategies are addressed in either a centralized or decentralized way. Bernstein and Federgruen (2004) develop oligopolistic inventory models with both price and service competition under three competing scenarios. Bernstein and Federgruen (2005) investigate the equilibrium behavior of decentralized supply chains with competing retailers under demand uncertainty. Gallego et al. (2005a) study a Bertrand oligopoly price competition game with general attraction demand functions and convex costs. This paper also proves a linear convergence of simultaneous discrete tatônnement scheme to the equilibrium.

We list here a couple of relevant references in the *game theory* literature. The differential version of our model also lives under the framework of *supermodular games*, which were introduced by Topkis (1979) and further generalized and extended by Milgrom and Roberts (1990). Vieille (2000) proves that two-player nonzero-sum stochastic games with finitely many states always have approximate equilibria. Osborne and Rubinstein (1994) cover basic concepts and theories in *game theory*, including *repeated games* that we will focus on in a later section.

1.3 Contribution and Outline

In the terminology of *game theory*, Perakis and Sood (2006) and Nguyen and Perakis (2005) consider a finite-horizon discrete-time stochastic game; Bernstein and Federgruen (2004) consider an infinite-horizon discrete-time stochastic game. To our knowledge, this paper is the first one to study a finite-horizon continuous-time stochastic game in RM literature. In terms of modeling demand uncertainty, Perakis and Sood (2006) and Nguyen and Perakis (2005) adapt ideas from *robust optimization* and assume that an uncertainty factor contained within an uncertainty set is associated with the demand function; Bernstein and Federgruen (2004) assume that the demand function is of the multiplicative form, i.e., the stochastic demand is the product of a deterministic demand rate and a general continuous random variable; Federgruen and Heching (1999) assume the additive form of the demand function in their numerical study. However, non-homogeneous Poisson processes do not fit into multiplicative or additive stochastic models. To our knowledge, this paper is the first one to utilize the non-homogeneous Poisson process to address demand uncertainty under competition in perishable asset revenue management. In summary, our model formulates the competitive dynamic pricing game of perishable assets as an intensity control game and complements previous works on *revenue management* and *oligopoly pricing*.

Previous works in RM on demand choice models without competition, such as Talluri and van Ryzin (2004a), Gallego et al. (2004) and van Ryzin and Liu (2004), assume prices of available products are given exogenously. This is consistent with current RM practice of separating pricing decisions from capacity allocation decisions. Presumably, one of the reasons to separate these two decisions is that the group responsible for pricing is looking into competitive issues in determining fares. These fares are then passed to the capacity allocation group. As explained in Gallego and van Ryzin (1994), capacity allocation can be interpreted as an attempt to adaptively synthesize optimal prices within the span of the fares given by the pricing group. While it is possible to analyze capacity allocation under fare restriction under competition, the computational burden of solving the resulting problem is daunting even for the single leg problem. Moreover, this does not resolve the problem of setting prices. Our model, in contrast, integrates pricing and capacity allocation and it is easier to analyze under competition even in a network setting. In practice, pricing teams can take our solution a step further and design a menu of efficient[3] fares that span the solution to

[3]By "efficient", we mean fares that are not dominated by convex combinations of other fares in terms of revenues

our competitive model.

In *game theory*, the best response strategy produces the most favorable immediate outcome for each player taking competitors' strategies as given. The best response strategy for players with abundant inventories[4] is to always price to maximize the revenue rate. The best response for players that do not have enough inventory to sustain *revenue maximizing prices* is to use the correspondingly higher *market clearing prices.* At such prices, inventories are expected to run out at the end of the sales horizon. In this paper, we show that a tatônnement scheme that results from competitors reacting with their best response functions converges to a Nash equilibrium[5].

An important question in games like these is what happens in equilibrium relative to other strategies in single stage games or finitely repeated games. To address this question we define *rational* players as those who always use their best response functions. Because of the convergence of the tatônment scheme we know that these players stick to their Nash equilibrium strategies. In contrast, *irrational* players deviate from the Nash equilibrium. Notice that rational players will converge to a Nash Equilibrium that is a best response to all other players including irrational players. Given this setup, we numerically investigate the effect that irrational players have on revenues, their own and those of other players.

Not surprisingly, irrational players consistently lose revenues relative to playing their equilibrium strategies. More interestingly, the losses incurred by irrational players themselves are consistently larger than the losses inflicted on other players. We find that rational players with more price-sensitive demands suffer larger losses when irrational players maximize their revenue rates instead of using their (higher) market clearing prices. However, by pricing low, irrational players run out of inventory before the end of the horizon allowing rational players to respond with higher prices. In some cases, rational players may come ahead as a result of irrational players' pricing low relative to their inventories.

When irrational players use their market clearing prices instead of their (higher) optimal revenue-rate maximizing prices, all rational players suffer with those who have more price-sensitive demands suffering the most. In this case irrational players do not run out of inventory until the end of the horizon so rational players have no way of recovering from the inflicted losses. In conclusion,

relative to capacity consumption.

[4]The game here refers to the nonzero-sum differential game formulated in §2.3.

[5]The Nash equilibrium here refers to an open-loop Nash equilibrium discussed in §3.

the negative effect on rational players is larger when irrational players use market clearing prices instead of their optimal revenue-rate maximizing prices. When irrational players use revenue-rate maximizing prices instead of optimal market clearing prices, the losses to rational players is much smaller and can in fact be negative.

When irrational players select random prices between their market clearing and revenue-rate maximizing prices, the effect can be either positive or negative on the rational players. Rational players with price-sensitive demands are more likely to suffer, but rational players with more inventories have more chances to recover and come out ahead after some irrational players run out of inventory.

The remainder of this paper is organized as follows: Section 2 describes our modeling assumptions and formulates both the nonzero-sum stochastic game and the corresponding nonzero-sum differential game. Section 3 characterizes the open-loop Nash Equilibrium for the differential game and presents sufficient conditions to guarantee its existence and uniqueness. Attraction Models, including MultiNomial Logit models, are studied as a special case and the robust stability of the open-loop Nash equilibrium with numerical experiments is also demonstrated in this section. Section 4 proves that the fixed-pricing heuristic suggested by the open-loop Nash equilibrium sustains a (relative) ϵ Nash equilibrium asymptotically. Section 5 studies both finitely and infinitely repeated versions of the single-stage game. Section 6 extends the basic problem to incorporate more complexities and features. Section 7 gives a few concluding remarks and points out some possible future works.

2 Assumptions and Formulations

2.1 Modeling Assumption and Notation

We consider a market for a single perishable asset with m competitors where demand is a function of the prices across this industry. More specifically, demand for the product is assumed to be a stochastic point process with Markovian intensities. At any time s, the vector of demand intensities $\lambda(s) = \big(\lambda_1(s), \lambda_2(s), \cdots, \lambda_m(s)\big)$ is determined by the current price vector $p(s) = \big(p_1(s), p_2(s), \cdots, p_m(s)\big)$ through a function $\lambda(p) : \mathbb{R}^m_+ \to \mathbb{R}^m_+$, i.e., $\lambda_i(s) = \lambda_i\big(p(s)\big)$ for $i = 1, 2, \cdots, m$. Thus, demand is a controlled Poisson process. We assume the demand intensity function $\lambda(p)$ is known and that it satisfies the following regularity conditions:

Assumption (A1). $\lambda(p)$ is continuous and differentiable, and

$$
\begin{aligned}
&(D1) \qquad && \frac{\partial \lambda_i}{\partial p_i} \leq 0\,, \quad \forall\, i\,, \\
&(D2) \qquad && \frac{\partial \lambda_i}{\partial p_j} \geq 0\,, \quad \forall\, j \neq i\,, \quad \forall\, i\,, \\
&(D3) \qquad && \sum_{j=1}^{m} \frac{\partial \lambda_i}{\partial p_j} < 0\,, \quad \forall\, i\,.
\end{aligned}
$$

These assumptions are fairly standard in *oligopoly pricing* literature. (D1) says that the demand intensity of a firm is decreasing in its own price. (D2) stipulates that an increase in the price of firm $j \neq i$ causes the demand intensity of firm i to increase. (D3) indicates that if all firms raise their prices, any firm will see a decrease in its sales. Assumption (D1)-(D3) is the "row-dominated by its diagonal" condition that is sufficient for the Jacobian matrix $\partial\lambda(p)/\partial p$ to be negative definite and non-singular in the interior of the domain, then by Inverse Function Theorem it can further ensure the inverse mapping $p(\lambda)$, $p(\cdot) : \mathbb{R}_+^m \to \mathbb{R}_+^m$ exists for demand intensity function $\lambda(p)$. Hence, there is a *one-to-one* correspondence between price vector $p = (p_1, p_2, \cdots, p_m)$ and demand intensity vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ under assumption (D1)-(D3).

Assumption (A2). Let $p_{-i} = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_m)$, $\forall\, i$, denote the pricing vector of the other $m-1$ firms who compete with firm i. For any given p_{-i}, $\forall\, i$, the inverse of $\lambda_i(p_i; p_{-i})$ is denoted by $p_i(\lambda_i; p_{-i})$ and the revenue rate $\hat{r}_i(\lambda_i; p_{-i}) := \lambda_i p_i(\lambda_i; p_{-i})$ is assumed to have a maximizer $\lambda_i^*(p_{-i}) = \{\lambda_i : r_i(\lambda_i; p_{-i}) = \max_{\lambda_i \geq 0} r_i(\lambda_i; p_{-i})\}$. We can also write the revenue rate in terms of price vector only as $r_i(p) = r_i(p_i; p_{-i}) := p_i \lambda_i(p_i; p_{-i})$. For any given p_{-i}, $\forall\, i$, the price p_i must be chosen from the set of allowable prices $\mathcal{P}_i(p_{-i}) = \big[0, p_i^\infty(p_{-i})\big]$ $\big(\mathbb{R}_+ \bigcup \{+\infty\}$ when $p_i^\infty(p_{-i}) = +\infty\big)$, where $p_i^\infty(p_{-i})$ is the *null* price for any given p_{-i} such that

$$
\lim_{p_i \to p_i^\infty(p_{-i})} \lambda_i(p) = \lim_{p_i \to p_i^\infty(p_{-i})} \lambda_i(p_i\,; p_{-i}) = 0 \;\text{ for any }\; p_{-i}\,,
$$

and

$$
\lim_{p_i \to p_i^\infty(p_{-i})} r_i(p) = \lim_{p_i \to p_i^\infty(p_{-i})} p_i \lambda_i(p_i\,; p_{-i}) = 0 \;\text{ for any }\; p_{-i}\,.
$$

The set of allowable intensity rates for firm i, $\forall\, i$ is denoted by $\Lambda_i(p_{-i}) = \{\lambda_i(p) : p_i \in \mathcal{P}_i(p_{-i})\}$.

Note that since $p_i^\infty(p_{-i}) \in \mathcal{P}_i(p_{-i})$, we always have $0 \in \Lambda_i(p_{-i})$, which guarantees that we can "turn off" the demand process when the firm runs out of items to sell.

We call a function $\lambda(p)$ that satisfies all of the assumptions above a *regular* demand intensity function, which is an extension of the single-variable *regular* demand function assumed in Gallego and van Ryzin (1994) to the multi-dimensional case.

2.2 Formulation of the Nonzero-sum Stochastic Game

We consider a finite-horizon, m-person, nonzero-sum, non-cooperative stochastic game of the optimal dynamic pricing problem considered in Gallego and van Ryzin (1994). The game is formulated as follows: At time zero, firm $i = 1, 2, \cdots, m$ has inventory $n_i \in \mathbb{Z}_+$ units of the same perishable asset and a finite time $t > 0$ to sell them. We assume the salvage value of the asset at time t is zero and that all other costs are sunk. The firms control the *regular* demand intensity by adjusting price[6]. At time $s \in [0, t]$, firm i applies its own non-anticipating price $p_i(s)$. Let N_s^i denote the number of items sold up to time s from firm $i = 1, 2, \cdots, m$. A demand for any firm i is realized at time s if $dN_s^i = 1$. We denote by $\mathcal{U}$ the joint Markovian allowable pricing policy space. Any joint allowable pricing policy $(p_1, p_2, \cdots, p_m) \in \mathcal{U}$ satisfies that for $\forall\, i$,

$$\int_0^t dN_s^i \le n_i\,, \quad \text{a.s.}$$

and

$$p_i(s) \in \mathcal{P}_i\big(p_{-i}(s)\big) \Longleftrightarrow \lambda_i(s) \in \Lambda_i\big(p_{-i}(s)\big)\,, \quad \forall\, s \in [0, t]\,.$$

By the Markovian property of $\mathcal{U}$, we mean the price policy $p_i(s)$ offered by any firm i is a function $p_i\big(s, n_i - N_s^i, p_{-i}(s)\big)$, i.e., a function of the elapsed time s, its own inventory level and other firms' prices at time s. In terms of *game theory*, we analyze strategies in feedback form, or in other words, closed-loop strategies. Here we interpret the Markovian property in this way rather than $p_i(s) = p_i\big(s, n_1 - N_s^1, n_2 - N_s^2, \cdots, n_m - N_s^m\big)$ because competitors' instant inventory levels might not be observable. However, if all firms are rational and play at the equilibrium, by monitoring the competitors' price, any firm can infer the instant inventory of other firms.

[6] As pointed out in Bernstein and Federgruen (2003, §4.2.), the price competition and demand/quantity competition end up with different Nash equilibrium. It is necessary to emphasize that we only consider price competition in this paper.

Given pricing policy $u \in \mathcal{U}$, initial stock level $n_i \in \mathbb{Z}_+$ and a sales horizon $t > 0$, we denote the expected profit for any firm i by

$$J_i = J_i(t, n_i, u) = J_i\big(t, n_i, u_1, u_2, \cdots, u_m\big) := \mathbb{E}\left[\int_0^t p_i(s)\, dN_s^i\right] .$$

The goal of each firm $i = 1, 2, \cdots, m$ is to maximize its total expected profit over $[0, t]$.

We assume all firms have perfect information of prices about each other and make decisions simultaneously. More specifically, any firm i observes its own inventory state $n_i - N_s^i$ and the price vector of all other firms $p_{-i}(s)$ at any time $s \in [0, t]$, and act upon that information. A set of policies $u^* = (u_1^*, u_2^*, \cdots, u_m^*) \in \mathcal{U}$ constitutes a *Nash equilibrium* if, whenever any firm modifies its policy away from the equilibrium, its own payoff will not increase. More precisely, u^* is called a *Nash equilibrium* if

$$J_i\big(t, n_i, u_i, u_{-i}^*\big) \le J_i\big(t, n_i, u_i^*, u_{-i}^*\big) , \quad \forall\, i ,$$

for any $(u_i, u_{-i}^*) = \big(u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_m^*\big) \in \mathcal{U}$. In other words, we are requiring that, for any $i = 1, 2, \cdots, m$, the policy u_i^* provides the optimal solution to the dynamic pricing problem for firm i while all firms $j \neq i$ use policy u_j^*. Generally it is extremely difficult to solve such a stochastic game for Nash equilibria and current research still stays at theoretically proving the existence of approximate equilibria.

2.3 Formulation of the Nonzero-sum Differential Game

Let us consider the following deterministic version of the stochastic game: At time zero, firm $i = 1, 2, \cdots, m$ has $x_i \in \mathbb{R}_+$ units of the same item and a finite time $t > 0$ to sell it. The instantaneous demand rate of firm $i = 1, 2, \cdots, m$ is *deterministic* and a function of the prices of all firms at time s, i.e., $p(s) = \big(p_1(s), p_2(s), \cdots, p_m(s)\big)$, again denoted by $\lambda_i(p(s))$. We assume $\lambda_i(\cdot), i = 1, 2, \cdots, m$ is a *regular* demand intensity function. As before, without loss of generality, we assume the salvage value of the product at time t is zero and that all other costs are sunk. We denote by $\mathcal{P}$ the deterministic allowable pricing policy space that is the deterministic counterpart of $\mathcal{U}$.

Given pricing policy $p = (p_1(s), p_2(s), \cdots, p_m(s)) \in \mathcal{P}$, initial stock level x_i and a sales horizon $t > 0$, we denote the total profit for firm i by

$$J_i^D = J_i^D(t, x_i, p) = J_i^D\big(t, x_i, p_1(s), p_2(s), \cdots, p_m(s)\big) := \int_0^t r_i\big(p(s)\big)\, ds\,, \tag{1}$$

where $r_i\big(p(s)\big) = p_i(s)\lambda_i\big(p_i(s); p_{-i}(s)\big)$. Each firm's problem is to maximize its own total revenue given by (1) by choosing an optimal pricing control path $p_i(s)$ for $s \in [0, t]$. We still consider the Markovian (or feedback) policy here in the sense of selecting controls according to the rule

$$p_i(s) = p_i\big(s, x_i(s), p_{-i}(s)\big)\,, \quad \text{where } x_i(s) = x_i - \int_0^s \lambda_i(p_i(u); p_{-i}(u))\, du\,.$$

This means each firm i observes current time t, its inventory position $x_i(t)$ and pricing policies $p_{-i}(t)$ adopted by its competitors, then chooses its own action as prescribed by the decision rule. An m-tuple $(p_1^*, p_2^*, \cdots, p_m^*) \in \mathcal{P}$ of policies constitutes a Nash equilibrium for the differential game if and only if for $\forall\, i = 1, 2, \cdots, m$ and $\forall\, (p_i, p_{-i}^*) = \big(p_1^*, \cdots, p_{i-1}^*, p_i, p_{i+1}^*, \cdots, p_m^*\big) \in \mathcal{P}$,

$$J_i^D\big(t, x_i, p_i, p_{-i}^*\big) \le J_i^D\big(t, x_i, p_i^*, p_{-i}^*\big)\,.$$

A Markovian or close-loop Nash equilibrium of the differential game can be found in principle through the Hamilton-Jacobi-Bellman equations, which is stated in the following proposition without proof.

Proposition 1. *Suppose that an m-tuple $\big(p_1^*, p_2^*, \cdots, p_m^*\big) \in \mathcal{P}$ of function $p_i^*(s, x_i) : [0, t] \times \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, \cdots, m$ is given and the following two conditions hold:*

(i) There is a unique, absolutely continuous solution $x(s) = \big(x_1(s), x_2(s), \cdots, x_m(s)\big) \in \mathbb{R}_+^m$ of the initial value problem

$$\dot{x}_i(s) = -\lambda_i\big(p_i^*(s); p_{-i}^*(s)\big)\,, \quad x_i(0) = x_i\,, \quad \forall\, i\,;$$

(ii) There exist continuously differentiable functions $V_i(s, x_i) : [0, t] \times \mathbb{R}_+ \to \mathbb{R}_+$, such that the

following Hamilton-Jacobi-Bellman equations are satisfied for all $(s, x_i) \in [0, t] \times \mathbb{R}_+$*:*

$$-\frac{\partial V_i(s, x_i)}{\partial s} = \max_{\lambda_i} \left[r_i\big(p_i^*(s); p_{-i}^*(s)\big) - \lambda_i(p_i^*(s); p_{-i}^*(s)) \frac{\partial V_i(s, x_i)}{\partial x_i} \right], \quad \forall\, i\,, \tag{2}$$

where $p_{-i}^*(s) = \big(p_1^*(s, x_1(s)), \cdots, p_{i-1}^*(s, x_{i-1}(s)), p_{i+1}^*(s, x_{i+1}(s)), \cdots, p_m^*(s, x_m(s))\big)$*, with boundary conditions*

$$V_i(t, x_i) = 0\,,\ \forall\, x_i > 0\,, \quad \textit{and} \quad V_i(s, 0) = 0\,,\ \forall\, s \in [0, t]\,, \quad \forall\, i\,, \tag{3}$$

then, if $p_i^*(s, x_i)$ *achieves the maximum of the Hamilton-Jacobi-Bellman equation (2), the* m*-tuple* $\big(p_1^*, p_2^*, \cdots, p_m^*\big) \in \mathcal{P}$ *is a Markovian Nash equilibrium of the differential game.*

3 Open-loop Nash Equilibrium of the Differential Game

We first consider the differential game with a *regular* demand intensity function defined in §2.1 and then treat some commonly used demand intensity functions later in §3.1-§3.3. It's not surprising that it is difficulty to solve partial differential equations (2)-(3). We will focus our attention on open-loop policies and identify the open-loop Nash equilibrium by using Hamiltonians. In order to proceed, let us standardize our model with constraints as the following objective functionals:

$$J_i^D\big(t, x_i, p; l_i\big) = \int_0^t r_i(p)\, ds + l_i\big(x_i - \int_0^t \lambda_i(p)\, ds\big)\,,$$

where l_i, $\forall\, i$ are Lagrangians, and the following dynamics:

$$\dot{x}_i(s) = -\lambda_i(p(s))\,, s \in [0, t]\,, \quad x_i(0) = x_i\,, \quad \forall\, i\,.$$

Lemma 1. *The open-loop Nash equilibrium of the nonzero-sum differential game, if exists, must be a fixed pricing policy.*

Proof. Let us define the Hamiltonians as $H_i(x, p, \mu_i) = r_i\big(p\big) - l_i\lambda_i(p) + \mu_i(-\lambda_i(p))$ for $\forall\, i$, where $\mu_i = \mu_i(s)$, $\forall\, i$ are the costate variables. Since the Hamiltonians are independent of the dynamics x and there is no salvage value, the costate equations and their transversality conditions are

$\dot{\mu_i}(s) = 0$, $\mu_i(t) = 0$, $\forall\, i$, which lead to $\mu_i(s) = 0$, $s \in [0,t]$, $\forall\, i$. Thus, the Hamiltonians become $H_i(x,p,\mu_i) = r_i(p) - l_i\lambda_i(p)$, $\forall\, i$, which are time-independent. Therefore, the open-loop Nash equilibrium $\big(p_1^*(s), p_2^*(s), \cdots, p_m^*(s)\big)$ as the maximizer of the Hamiltonians with respect to p is time-independent and has to be a fixed pricing policy for any firm i. □

Theorem 1. *Consider the nonzero-sum differential game. If $p^* = \big(p_1^*, p_2^*, \cdots, p_m^*\big)$ is a solution to the following system of equations*

$$\begin{cases} \frac{x_i}{t} = \lambda_i(p)\,, & i \in S\,, \\ \lambda_i^*(p_{-i}) = \lambda_i(p)\,, & i \in \overline{S}\,, \end{cases} \tag{4}$$

for some $S \subseteq \{1, 2, \cdots, m\}$ and

$$\begin{cases} \frac{x_i}{t} < \lambda_i^*(p_{-i}^*)\,, & i \in S\,, \\ \frac{x_i}{t} \geq \lambda_i^*(p_{-i}^*)\,, & i \in \overline{S}\,, \end{cases} \tag{5}$$

then the fixed pricing policy p^* is an open-loop Nash equilibrium for the initial condition t and x.

Proof. By Lemma 1, we only need to scrutinize among the fixed pricing policies for the open-loop Nash equilibrium. By Gallego and van Ryzin (1994, Prop.2), for any firm i, the optimal policy to the monopoly fluid model with all the other firms applying fixed pricing policies p_{-i} is $\lambda_i^* = \min\{\lambda_i^*(p_{-i}), x_i/t\}$. Suppose S is the index set such that λ_i^* is equal to x_i/t, i.e., $S = \{i : \lambda_i^* = x_i/t\} \subseteq \{1, 2, \cdots, m\}$, and $\overline{S}$ is the index set such that λ_i^* is equal to the maximizer $\lambda_i^*(p_{-i})$ of function $r_i(\lambda_i; p_{-i})$ with respect to λ_i. Also note that $\overline{S}$ and S correspond to the cases of Lagrangian $l_i = 0$ and $l_i > 0$, respectively. Then by the demand and price relationship, we have a system (4) of m equations and m unknowns to characterize the open-loop Nash equilibrium while assumption (5) on the initial condition ensures $\lambda_i^* = \min\{\lambda_i^*(p_{-i}^*), x_i/t\}$. □

Note that S could be the empty set or the whole set and for each case, the system of equations (4) becomes $\lambda_i^*(p_{-i}) = \lambda_i(p), i = 1, 2, \cdots, m$ or $x_i/t = \lambda_i(p), i = 1, 2, \cdots, m$, respectively. If demand intensity $\lambda_i(p)$ violates the continuity assumption at $\lambda_i^{-1}(x_i/t; p_{-i})$, firm i could use multiple fixed prices in equilibrium but the sum of demand intensities at different prices is still required to be x_i/t. If the revenue rate $r_i(\lambda_i; p_{-i})$ has more than one maximizer, there exist multiple open-loop

Nash equilibria.

Corollary 1. *Consider the nonzero-sum differential game. Suppose for $\forall\, i$, $r_i(\lambda_i; p_{-i})$ is concave or log-concave in λ_i for given fixed p_{-i}. A fixed pricing policy $p^* = \left(p_1^*, p_2^*, \cdots, p_m^*\right)$ is an open-loop Nash equilibrium for the initial condition t and x if and only if p^* is a solution to the following system of equations*

$$\begin{cases} \frac{x_i}{t} = \lambda_i(p)\,, & i \in S\,, \\ \lambda_i(p) + p_i \frac{\partial \lambda_i(p)}{\partial p_i} = 0\,, & i \in \overline{S}\,, \end{cases} \tag{6}$$

for some $S \subseteq \{1, 2, \cdots, m\}$ and

$$\begin{cases} \frac{x_i}{t} < -p_i \frac{\partial \lambda_i(p)}{\partial p_i}\big|_{p^*}\,, & i \in S\,, \\ \frac{x_i}{t} \geq -p_i \frac{\partial \lambda_i(p)}{\partial p_i}\big|_{p^*}\,, & i \in \overline{S}\,. \end{cases} \tag{7}$$

Proof. Since $r_i(\lambda_i; p_{-i})$ is concave or log-concave in λ_i for $\forall\, i$, system of equations (4) is equivalent to

$$\begin{cases} \frac{x_i}{t} = \lambda_i(p)\,, & i \in S\,, \\ \frac{\partial r_i(\lambda_i; p_{-i})}{\partial \lambda_i}\big|_{\lambda_i = \lambda_i(p)} = \frac{\partial p_i(\lambda_i; p_{-i})}{\partial \lambda_i}\lambda_i + p_i(\lambda_i; p_{-i})\big|_{\lambda_i = \lambda_i(p)} = 0\,, & i \in \overline{S}\,. \end{cases} \tag{8}$$

By the Implicit Function Theorem,

$$\frac{\partial p_i(\lambda_i; p_{-i})}{\partial \lambda_i} = -\frac{1}{-\frac{\partial \lambda_i}{\partial p_i}} = \frac{1}{\frac{\partial \lambda_i}{\partial p_i}} < 0\,.$$

Thus, system of equations (6) is equivalent to (8), and consequently equivalent to (4). Hence by Theorem 1, if p^* is a solution to the equations set (6) for some set S, then the fixed pricing policy p^* is an open-loop Nash equilibrium for the nonzero-sum differential game, while condition (7) guarantees

$$\lambda_i^* = \min\left\{ -p_i \frac{\partial \lambda_i(p)}{\partial p_i}\bigg|_{p^*}, \frac{x_i}{t} \right\} = \min\{\lambda_i^*(p_{-i}^*), \frac{x_i}{t}\}\,, \quad \forall\, i\,.$$

The other direction of the claim can be easily obtained by directly checking the definition of open-loop Nash equilibrium. $\square$

In Corollary 1, the concavity or log-concavity of the revenue rate function in λ_i can be relaxed to quasi-concavity together with vanishing limit on boundary of the range that would also guarantee the maximizer of the function is characterized by the first-order condition.

To see if some fixed pricing policy p is an open-loop Nash equilibrium, Corollary 1 gives a criterion (6) that is only in terms of the demand intensity functions, however, Theorem 1, as well as the construction of open-loop Nash equilibria demonstrated later in Example 1, is made by looking at the maximizer $\lambda_i^*(p_{-i})$ of the revenue rate function. Criterion (6) and (4) are equivalent but provide observations from different angles: criterion (6) is easy-to-check but requires additional concavity or log-concavity assumption while criterion (4) is more general and intuitive. These two characterization results of the open-loop Nash equilibrium basically say that those firms with index i in set S have limited inventories and can afford to price higher at the market clearing price (the price corresponding to the run-out rate x_i/t); those firms with index in set $\overline{S}$ have abundant inventories and are at the prices to maximize their revenue rates.

Example 1. To get insight into Theorem 1, we consider the following demand intensity function, which is considered in Christen (2005):

$$\lambda_1 = a - b(p_1 - \gamma p_2)\,, \quad \lambda_2 = a - b(p_2 - \gamma p_1)\,, \quad \text{where } 0 \leq \gamma < 1\,. \tag{9}$$

The parameter γ is equal to the ratio of cross-price to own-price effects and captures the degree of competition between firms (see Vives (1999); Varian (1992)). We assume $\gamma \geq 0$ because we focus on noncomplementary products. To ensure that total demand $\lambda_1 + \lambda_2$, does not increase with prices, we assume $\gamma < 1$. The parameter b indicates demands' price sensitivity. Such a demand system can be derived as the first-order condition of a representative buyer's maximization of an appropriately defined concave utility function (see Vives (1984)). Notice that demand intensity functions (9) do satisfy assumptions (D1)-(D3).

From (9), We have the following revenue rate functions: $r_1(\lambda_1; p_2) = (a/b + \gamma p_2)\lambda_1 - {\lambda_1}^2/b$, $r_2(\lambda_2; p_1) = (a/b+\gamma p_1)\lambda_2 - {\lambda_2}^2/b$. The maximizers of the revenue rates are: $\lambda_1^*(p_2) = (a+\gamma b p_2)/2$, $\lambda_2^*(p_1) = (a + \gamma b p_1)/2$. By Theorem 1, we know that

case (i) $S = \emptyset$, if $x_1/t \geq a/(2-\gamma)$, $x_2/t \geq a/(2-\gamma)$, then fixed pricing policy $\big(p_1^*(s), p_2^*(s)\big) = \big(a/[b(2-\gamma)], a/[b(2-\gamma)]\big)$ corresponding to $\big(\lambda_1^*(s), \lambda_2^*(s)\big) = \big(a/(2-\gamma), a/(2-\gamma)\big)$, constitutes a unique open-loop Nash equilibrium;

case (ii) $S = \{1\}$, if $x_1/t < a/(2-\gamma)$, $\gamma x_1/t + (2-\gamma^2)x_2/t \geq a(1+\gamma)$, then fixed pricing policy $\big(p_1^*(s), p_2^*(s)\big) = \big(\frac{a(2+\gamma)-2\,x_1/t}{b(2-\gamma^2)}, \frac{a(1+\gamma)-\gamma\,x_1/t}{b(2-\gamma^2)}\big)$ corresponding to $\big(\lambda_1^*(s), \lambda_2^*(s)\big) = \big(x_1/t, \frac{a(1+\gamma)-\gamma\,x_1/t}{2-\gamma^2}\big)$, constitutes a unique open-loop Nash equilibrium;

case (iii) $S = \{2\}$, if $x_2/t < a/(2-\gamma)$, $\quad (2-\gamma^2)x_1/t + \gamma x_2/t \geq a(1+\gamma)$, then fixed pricing policy $\big(p_1^*(s), p_2^*(s)\big) = \big(\frac{a(1+\gamma)-\gamma\, x_2/t}{b(2-\gamma^2)}, \frac{a(2+\gamma)-2\, x_2/t}{b(2-\gamma^2)}\big)$ corresponding to $\big(\lambda_1^*(s), \lambda_2^*(s)\big) = \big(\frac{a(1+\gamma)-\gamma\, x_2/t}{2-\gamma^2}, x_2/t\big)$, constitutes a unique open-loop Nash equilibrium;

case (iv) $S = \{1,2\}$, if $\gamma x_1/t + (2-\gamma^2)x_2/t < a(1+\gamma)$, $\quad (2-\gamma^2)x_1/t + \gamma x_2/t < a(1+\gamma)$, then fixed pricing policy $\big(p_1^*(s), p_2^*(s)\big) = \big(\frac{a(1+\gamma)-x_1/t-\gamma\, x_2/t}{b(1-\gamma^2)}, \frac{a(1+\gamma)-\gamma\, x_1/t-x_2/t}{b(1-\gamma^2)}\big)$ corresponding to $\big(\lambda_1^*(s), \lambda_2^*(s)\big) = \big(x_1/t, x_2/t\big)$, constitutes a unique open-loop Nash equilibrium.

Figure 1 illustrates that the reduced initial state space $\mathbb{R}_+^2$ of $(x_1/t, x_2/t)$ is divided into four "subspaces" that correspond to four cases we have discussed above.

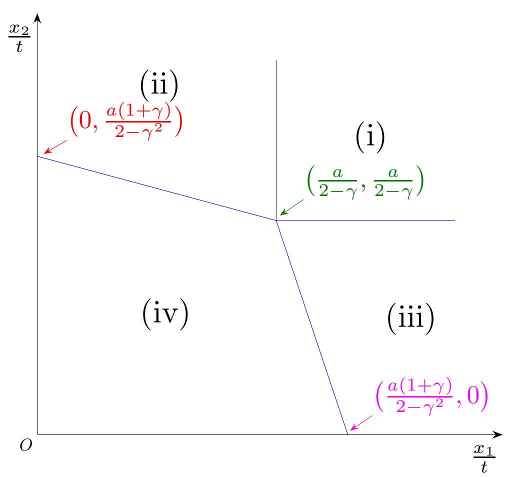

Figure 1: The Reduced Initial State Space for a Two-dimensional Linear Demand Function

Theorem 2. *If the demand intensity functions $\lambda_i(p), i = 1, 2, \cdots, m$, are twice continuously differentiable and the following condition*

$$-\frac{\partial \lambda_i}{\partial p_i}\frac{\partial \lambda_i}{\partial p_j} + \lambda_i \frac{\partial^2 \lambda_i}{\partial p_i \partial p_j} \geq 0 \tag{10}$$

holds for all $i \neq j$, $i, j = 1, 2, \cdots, m$, then the nonzero-sum differential game has at least one open-loop Nash equilibrium and the set of Nash equilibriais a lattice. Under the additional condition

$$\left(\frac{\partial \lambda_i}{\partial p_i}\right)^2 - \lambda_i \frac{\partial^2 \lambda_i}{\partial {p_i}^2} + \left(\frac{\lambda_i}{p_i}\right)^2 > \sum_{j \neq i} -\frac{\partial \lambda_i}{\partial p_i}\frac{\partial \lambda_i}{\partial p_j} + \lambda_i \frac{\partial^2 \lambda_i}{\partial p_i \partial p_j}, \quad \forall\, i, \tag{11}$$

the open-loop Nash equilibrium is unique.

Proof. We show that $r_i(p), \forall\, i$ are log-supermodular in p under condition (10). Let $\tilde{r}_i(p) = \log r_i(p)$ and $\tilde{\lambda}_i(p) = \log \lambda_i(p)$. Since $\lambda_i(p), i = 1, 2, \cdots, m$, are twice continuously differentiable, condition (10) is equivalent to supermodularity of $\tilde{r}_i(p) = \log p_i + \tilde{\lambda}_i(p)$ in p since

$$\frac{\partial^2 \tilde{r}_i(p)}{\partial p_i \partial p_j} = \frac{\partial^2 \tilde{\lambda}_i(p)}{\partial p_i \partial p_j} = \frac{1}{{\lambda_i}^2}\left[-\frac{\partial \lambda_i}{\partial p_i}\frac{\partial \lambda_i}{\partial p_j} + \lambda_i \frac{\partial^2 \lambda_i}{\partial p_i \partial p_j}\right] \geq 0\,, \quad j \neq i\,,\; i = 1, 2, \cdots, m\,. \tag{12}$$

Under any initial condition of t and x, suppose $S = \{\, i \,|\, \lambda_i(p) = x_i/t\} \subseteq \{1, 2, \cdots, m\}$, then we have

$$\frac{\partial^2 \tilde{r}_i(p)}{\partial p_i \partial p_j} = 0\,, \quad j \neq i\,,\; i \in S\,, \tag{13}$$

while condition (10) guarantees

$$\frac{\partial^2 \tilde{r}_i(p)}{\partial p_i \partial p_j} \geq 0\,, \quad j \neq i\,,\; i \in \overline{S}\,. \tag{14}$$

Thus, under any initial condition of t and x, combining (13) and (14), we have a nonzero-sum log-supermodular game. By Milgrom and Roberts (1990, Theorem 5), supermodularity is a sufficient condition for the existence of a lattice of Nash equilibria in the log-transformed game, which guarantees the existence of a lattice of Nash equilibria in the original game since log-transform is a monotone transformation. Moreover, condition (11) is equivalent to

$$-\frac{\partial^2 \tilde{r}_i(p)}{\partial {p_i}^2} > \sum_{j \neq i} \frac{\partial^2 \tilde{r}_i(p)}{\partial p_i \partial p_j}\,, \quad i = 1, 2, \cdots, m\,, \tag{15}$$

which ensures

$$-\frac{\partial^2 \tilde{r}_i(p)}{\partial {p_i}^2} > \sum_{j \neq i} \frac{\partial^2 \tilde{r}_i(p)}{\partial p_i \partial p_j}\,, \quad i \in \overline{S}\,.$$

Hence, by Milgrom and Roberts (1990), we have the uniqueness of the Nash equilibrium in the log-transformed game, thus in the original game. □

Theorem 2 gives sufficient conditions for the existence and uniqueness of the open-loop Nash equilibrium under all possible initial conditions of t and x. From the proof, we can see that for a specific initial state, in order to guarantee the existence and uniqueness, we need condition

(10) and (11) to hold only for some subset $S \subseteq \{1, 2, \cdots, m\}$. Considering the log-transformed game is a technique to reduce complexity and simplify notation while conserving supermodularity since log-transformation is an order-preserving map of the strategy space that does not affect supermodularity. Furthermore, condition (15) is the so-called "diagonal dominance" condition that ensures the uniqueness of Nash equilibrium for supermodular games. These sufficient conditions are essentially conditions to guarantee the existence and uniqueness of Nash equilibrium in the static supermodular game where the payoff function of each player is the revenue rate function and there is no inventory effect involved.

Condition (10) has an intuitive and precise interpretation: firm i's price elasticity of demand is increasing in the rivals' prices. Condition (11) is interpreted in the way that if it confines the allowable pricing policy space, then the uniqueness result holds only under that confinement. By combining (12) and (15), we see that condition (10) and (11) can guarantee $\partial^2 \tilde{r}_i(p)/\partial {p_i}^2 < 0$ thus Corollary 1 holds as a result. Both condition (10) and (11) in terms of only demand intensity functions are not universal but most commonly used demand functions would satisfy under mild conditions on the parameters (see Vives (1999, §2.5) for two more sufficient conditions of uniqueness that are less easy to be verified). Next, we consider several most frequently used classes of demand functions and show that the existence and uniqueness of the open-loop Nash equilibrium for the differential game can be guaranteed only by assumptions (D1)-(D3) in most cases.

Under the existence and uniqueness sufficient conditions, there is a one-to-one correspondence between an initial condition and an open-loop Nash equilibrium under that initial condition. Then we can write the open-loop Nash equilibrium p^* as a function of the initial condition t and x, i.e., $p^* = p^*(t, x)$. We have the following results discussing the properties of the open-loop Nash equilibrium p^* as a function of the initial state (t, x).

Corollary 2. *If $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$ is a solution to the system of equations $\lambda_i^*(p_{-i}) = \lambda_i(p)$, $i = 1, 2, \cdots, m$, then p^* is the open-loop Nash equilibrium if the initial state condition satisfies $x_i/t \geq \lambda_i^*(p_{-i}^*)$, $i = 1, 2, \cdots, m$.*

Corollary 2 is nothing but a special case of Theorem 1 and we list it here as a corollary to emphasize the observation that if all the firms have large initial inventories, the differential game reduces to a static pricing game generating a fixed pricing policy independent of the initial condition.

Corollary 3. *Suppose condition (10) and (11) hold. The open-loop Nash equilibrium* $p^*(t,x)$: $\mathbb{R}_+^{m+1} \to \mathbb{R}_+^m$ *that is a solution to the system of equations (4) is a function in a lower dimension domain* $\mathbb{R}_+^m$*, and more specifically,* $p^*(t,x) = p^*(f(t,x))$*, where* $f : \mathbb{R}_+^{m+1} \to \mathbb{R}_+^m, (t, x_1, x_2, \cdots, x_m) \mapsto (x_1/t, x_2/t, \cdots, x_m/t)$.

Corollary 3 essentially argues that the initial state (t,x) plays a role in determining the open-loop Nash equilibrium only in terms of the run-out rates x_i/t, $i = 1, 2, \cdots, m$.

Corollary 4. *If the demand intensity functions* $\lambda_i(p)$, $\forall\, i$ *are twice continuously differentiable and condition (10), (11) hold, then the open-loop Nash equilibrium* $p^*(t,x) : \mathbb{R}_+^{m+1} \to \mathbb{R}_+^m$ *that is a solution to the system of equations (4), is a continuous function in* t *and* x.

Proof. Since the demand intensity functions $\lambda_i(p)$, $\forall\, i$ are twice continuously differentiable, the revenue rate maximizers $\lambda_i^*(p_{-i})$, $\forall\, i$ are at least differentiable. For any two different subsets of $\{1, 2, \cdots, m\}$, conditions (5) are mutually exclusive and there are $m!$ different subsets of $\{1, 2, \cdots, m\}$ in total, thus the initial state space $\mathbb{R}_+^{m+1}$ of $(t, x_1, x_2, \cdots, x_m)$ is divided into $m!$ "subspaces". By the Inverse Function Theorem, the solution $p^*(t,x)$ to the system of equations (4) is differentiable within the interior of each "subspace"; and it is also continuous on the boundary of two "subspaces" in the initial state space. □

Corollary 4 demonstrates the continuity property of the open-loop Nash equilibrium, which implies robustness of the equilibrium in the sense that small perturbations of the initial state due to possible data inaccuracy will not change the equilibrium too much.

We defer the statement and proof of monotone property that $p^*(t,x)$ is decreasing in x/t to §3.4. Corollary 9 shows that the whole equilibrium price vector drops as the inventory of some firm increases and/or sales horizon shortens with the initial stock levels of all other firms fixed.

3.1 General Attraction Models

In the attraction models, the demand intensity function is a market share that is proportional to an attraction value among a fixed total potential market size. Specifically, the market share m_i of firm i equals its marketing effort quantitatively measured by attraction value a_i divided by total marketing effort, i.e.,

$$m_i = \frac{a_i}{a_0 + \sum_{j=1}^{m} a_j}\,, \quad \forall\, i\,, \tag{16}$$

where a_0 is the constant value of the no-purchase option. Bell et al. (1975) demonstrate that if four "axiom" assumptions are satisfied and the competitive situation is completely determined by the attraction vector $a = (a_0, a_1, \cdots, a_m)$, then the market share function has to be of the form (16). In that paper, the market share is actually assumed to be a *norm* of the attraction vector. Three of these assumptions are related to the definition of *norm* and the other one is natural and intuitive. Due to this generality, with the generalized MultiNomial Logit (MNL) models as a special case, attraction models are among the most commonly applied market share models. In standard marketing literature, the attraction values a may depend on the price vector and/or other characteristics such as quality, advertising efforts and service levels. In this section, we will restrict the attraction value a_i of any firm i, $i = 1, 2, \cdots, m$ as a general, twice continuously differentiable nonnegative function of its own price, i.e., $a_i = a_i(p_i) \geq 0$. Later we will generalize attraction values to embrace item quality or service level q_i. If the fixed total market size is M, we have the following demand intensity functions:

$$\lambda_i(p) = M \frac{a_i(p_i)}{\sum_{j=0}^{m} a_j(p_j)}\,, \quad \forall\, i\,, \tag{17}$$

where $a_0(p_0)$ is interpreted as the no-purchase option value $a_0 \geq 0$.

Corollary 5. *Consider the general attraction model with demand intensity function (17). Suppose condition (10) and (11) hold, then if $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$ is a solution to the following system of equations*

$$\begin{cases} \frac{x_i}{t} = M \frac{a_i}{\sum_{j=0}^{m} a_j}\,, & i \in S \\ a_i + p_i \frac{\partial a_i}{\partial p_i} \sum_{j \neq i} a_j = 0\,, & i \in \overline{S}\,, \end{cases}$$

for some $S \subseteq \{1, 2, \cdots, m\}$ and

$$\begin{cases} \frac{x_i}{t} < \frac{M \frac{\partial a_i}{\partial p_i} \sum_{j \neq i} a_j}{(\sum_{j=0}^{m} a_j)^2}\Big|_{p^*}\,, & i \in S\,, \\ \frac{x_i}{t} \geq \frac{M \frac{\partial a_i}{\partial p_i} \sum_{j \neq i} a_j}{(\sum_{j=0}^{m} a_j)^2}\Big|_{p^*}\,, & i \in \overline{S}\,, \end{cases}$$

the fixed pricing policy p^* is the unique open-loop Nash equilibrium for the initial condition t and

x.

Corollary 5 is a particular version of Theorem 1 for attraction models and is written in terms of attraction values. As we can see above, we cannot write conditions that the equilibrium should satisfy in a nice and neat form, but for the existence and uniqueness of the equilibrium we do have a simplified version of Theorem 2 only in terms of attraction values. A similar result has been demonstrated by Bernstein and Federgruen (2004) in an infinite-horizon game context.

Lemma 2. *Consider the general attraction model with demand intensity function (17). We have that assumptions (D1)-(D3) on demand intensity functions are equivalent to the following set of assumptions on the attraction value functions:*

$$(A1) \qquad \frac{\partial a_i}{\partial p_i} \leq 0\,, \quad i = 1, 2, \cdots, m\,,$$

$$(A2) \qquad \frac{a_i}{\sum_{j \neq i} a_j} < \frac{\frac{\partial a_i}{\partial p_i}}{\sum_{j \neq i} \frac{\partial a_j}{\partial p_j}}\,, \quad i = 1, 2, \cdots, m\,.$$

Proof. By Bernstein and Federgruen (2004, Lemma 2), we have

$$\frac{\partial \lambda_i}{\partial p_i} = \frac{\partial \tilde{a}_i}{\partial p_i} \lambda_i (1 - \frac{\lambda_i}{M})\,, \forall\, i\,, \qquad \frac{\partial \lambda_i}{\partial p_j} = -\frac{\partial \tilde{a}_j}{\partial p_j} \frac{\lambda_i \lambda_j}{M}\,, \forall\, j \neq i\,, \tag{18}$$

where $\tilde{a}_i = \log a_i$. Thus, assumptions (D1) and (D2) are equivalent to $\frac{\partial \tilde{a}_i}{\partial p_i} = \frac{1}{a_i} \frac{\partial a_i}{\partial p_i} \leq 0$ for all i, and consequently equivalent to assumption (A1).

By taking derivatives directly in terms of attraction values, we have

$$\frac{\partial \lambda_i}{\partial p_i} = \frac{M}{(\sum_{j=0}^m a_j)^2} \frac{\partial a_i}{\partial p_i} \sum_{j \neq i} a_j\,, \forall\, i\,, \qquad \frac{\partial \lambda_i}{\partial p_j} = -\frac{M}{(\sum_{j=0}^m a_j)^2}\, a_i \frac{\partial a_j}{\partial p_j}\,, \forall\, j \neq i\,.$$

Thus, assumption (D3) is equivalent to

$$\frac{(\sum_{j=0}^m a_j)^2}{M} \sum_{j=0}^m \frac{\partial \lambda_i}{\partial p_j} = \frac{\partial a_i}{\partial p_i} \sum_{j \neq i} a_j - a_i \sum_{j \neq i} \frac{\partial a_j}{\partial p_j} < 0\,, \quad \forall\, i\,,$$

and consequently equivalent to assumption (A2) if (A1) holds.

In summary, for the general attraction model, we have the following detailed equivalence relationship between assumptions in terms of demand function and conditions in terms of attraction values: $(D1) \Longleftrightarrow (A1)\,, \quad (D2) \Longleftrightarrow (A1)\,, \quad (D1)+(D3) \Longleftrightarrow (A1)+(A2)\,.$ $\square$

Theorem 3. *Consider the differential game for the general attraction model with demand intensity function (17). For any initial condition t and x,*

(a) if assumption (A1) $\big($or equivalently (D1)$\big)$ is satisfied, there is a lattice of open-loop Nash equilibria;

(b) if $\tilde{a}_i$ is concave in p_i for any $i = 1, 2, \cdots, m$ and assumptions (A1), (A2) $\big($or equivalently (D1)-(D3)$\big)$ are satisfied, there is a unique open-loop Nash equilibrium.

Proof. (a) We show that the revenue rate functions $r_i(p)$, $\forall\, i$ are log-supermodular in p under assumption (A1). Since $\tilde{r}_i(p) := \log r_i(p) = \log p_i + \log \lambda_i(p) = \log p_i + \tilde{\lambda}_i(p)$ and the twice continuous differentiability of $r_i(p)$, we only need to check $\partial^2 \tilde{r}_i(p)/\partial p_i \partial p_j = \partial^2 \tilde{\lambda}_i(p)/\partial p_i \partial p_j \geq 0$. By (18), we have $\partial\tilde{\lambda}_i/\partial p_i = (1/\lambda_i)(\partial\lambda_i/\partial p_i) = (\partial\tilde{a}_i/\partial p_i)(1-\lambda_i/M)$, thus by assumption (A1), $\partial^2\tilde{\lambda}_i/\partial p_i\partial p_j = (\partial\tilde{a}_i/\partial p_i)(\partial\tilde{a}_j/\partial p_j)(\lambda_i/M)(\lambda_j/M) \geq 0$. By the same argument in Theorem 2, the differential game is log-supermodular, no matter what the initial state is, hence there is a lattice of Nash equilibria.

(b) By Theorem 2, we only need to verify (15). By (18), we have $\partial\tilde{r}_i/\partial p_i = 1/p_i + \partial\tilde{\lambda}_i/\partial p_i$ and thus

$$\frac{\partial^2 \tilde{r}_i}{\partial {p_i}^2} = -\frac{1}{{p_i}^2} + \frac{\partial^2 \tilde{a}_i}{\partial {p_i}^2}(1-\frac{\lambda_i}{M}) - (\frac{\partial \tilde{a}_i}{\partial p_i})^2 \frac{\lambda_i}{M}(1-\frac{\lambda_i}{M}) < -(\frac{\partial \tilde{a}_i}{\partial p_i})^2 \frac{\lambda_i}{M}(1-\frac{\lambda_i}{M})\,, \tag{19}$$

since $\tilde{a}_i$ is concave in p_i. By assumption (A2), we have

$$(1-\frac{\lambda_i}{M})\frac{\partial \tilde{a}_i}{\partial p_i} = (1-\frac{\lambda_i}{M})\frac{\frac{\partial a_i}{\partial p_i}}{a_i} < (1-\frac{\lambda_i}{M})\frac{\sum_{j\neq i}\frac{\partial a_j}{\partial p_j}}{\sum_{j\neq i} a_j} = \frac{\sum_{j\neq i} a_j}{\sum_{j=0}^{m} a_j}\frac{\sum_{j\neq i}\frac{\partial a_j}{\partial p_j}}{\sum_{j\neq i} a_j} = \sum_{j\neq i}\frac{\partial \tilde{a}_j}{\partial p_j}\frac{\lambda_j}{M}\,. \tag{20}$$

Hence, combining (19) and (20), we get

$$-\frac{\partial^2 \tilde{r}_i}{\partial {p_i}^2} > (\frac{\partial \tilde{a}_i}{\partial p_i})^2\frac{\lambda_i}{M}(1-\frac{\lambda_i}{M}) > \frac{\partial \tilde{a}_i}{\partial p_i}\frac{\lambda_i}{M}\sum_{j\neq i}\frac{\partial \tilde{a}_j}{\partial p_j}\frac{\lambda_j}{M} = \sum_{j\neq i}\frac{\partial^2 \tilde{\lambda}_i}{\partial p_i \partial p_j} = \sum_{j\neq i}\frac{\partial^2 \tilde{r}_i}{\partial p_i \partial p_j}\,, \quad \forall\, i\,.$$

$\square$

By taking a close look at inequalities (19) and (20), we can relax "$<$" in assumption (A2) $\big($or equivalently (D3)$\big)$ to "$\leq$" and the result still holds if inequality (19) holds strictly.

The MNL model assumes a specific structure of the attraction value functions as

$$a_i(p_i) = \beta_i \exp\{-\alpha_i p_i\}\,, \quad \alpha_i, \beta_i > 0\,,\; i = 1, 2, \cdots, m\,. \tag{21}$$

Obviously, $\tilde{a}_i$ is linear and hence concave in p_i, and assumption (A2) holds weakly for the Logit model when $\alpha_i = \alpha$ for $\forall\, i$. Hence for the MNL model with demand intensity function (21), there is a unique open-loop Nash equilibrium when $\alpha_i = \alpha$ for $\forall\, i$, for any initial condition t and x. From the perspective of solving the system of equations $\partial r_i(p)/\partial p_i = 0$, $\forall\, i$ for a unique solution, we can get a weaker sufficient condition for the MNL model to guarantee the uniqueness that doesn't require α_i, $\forall\, i$ to be the same. We state the result in the following as a proposition without proof (see Gallego et al. (2005a) for a detailed treatment to general attraction models from the alternative perspective).

Proposition 2. *Consider the differential game for the MNL model with demand intensity function (21). For any initial condition t and x, there is a unique open-loop Nash equilibrium.*

The most commonly used "Multiplicative Competitive Interaction" structure takes the form of attraction values as

$$a_i(p_i) = \alpha_i {p_i}^{-\beta_i}\,, \quad \alpha_i, \beta_i > 0\,,\; i = 1, 2, \cdots, m\,. \tag{22}$$

However, $\tilde{a}_i$ is convex in p_i, so we need to derive conditions to guarantee the uniqueness of the open-loop Nash equilibrium for those classes of models with convex but "not too convex" attraction values.

Corollary 6. *Consider the differential game for the general attraction model with demand intensity function (17). For any initial condition, if assumptions (A1), (A2) (or equivalently (D1)-(D3)) are satisfied, there is a unique open-loop Nash equilibrium under the following additional condition:*

$$\frac{\partial^2 \tilde{a}_i}{\partial {p_i}^2}\left(1 - \frac{\lambda_i}{M}\right) \geq \frac{1}{{p_i}^2}\,, \quad i = 1, 2, \cdots, m\,. \tag{23}$$

Proof. Condition (23) guarantees inequality (19) to hold weakly. □

In order to accommodate more general classes of models with "not too convex" attraction values, Corollary 6 gives a sufficient condition on $\tilde{a}_i$ that does not require it to be concave. As an immediate result, for the MCI model with demand intensity function (22), there is a unique open-loop Nash equilibrium under additional condition $\beta_i(1 - \lambda_i/M) \geq 1$, $\forall\, i$. But from the alternative perspective of directly checking the system of equations $\partial r_i(p)/\partial p_i = 0$, $\forall\, i$, a weaker sufficient condition can be obtained.

Proposition 3. *Consider the differential game for the MCI model with demand intensity function (22). For any initial condition t and x, there is a unique open-loop Nash equilibrium when $\beta_i > 1$, $\forall\, i$.*

3.2 The Linear Model

The demand intensity function in the linear model has the form of

$$\lambda_i(p) = a_i - b_i p_i + \sum_{j \neq i} c_{ij} p_j \,, \quad a_i, b_i > 0 \,, c_{ij} \geq 0 \,, j \neq i \,, i = 1, 2, \cdots, m \,. \tag{24}$$

It is easy to check that the sufficient condition (10) for the existence of the open-loop Nash equilibrium is satisfied for the linear model due to the nonnegativity of the parameters. Assumption (D3) requires

$$b_i > \sum_{j \neq i} c_{ij} \,, \quad i = 1, 2, \cdots, m \,, \tag{25}$$

which guarantees matrix $T_S = (t_{ij})_S$ to be substochastic, where $t_{ij} = c_{ij}/(2b_i)$ for $\forall\, j \neq i$, $t_{ii} = 1/2$ for $i \in S$ and $t_{ii} = 0$ for $i \in \overline{S}$, since $1/2 + \sum_{j \neq i} c_{ij}/2b_i < 1$ for $i \in S$ and $\sum_{j \neq i} c_{ij}/2b_i < 1$ for $i \in \overline{S}$.

Corollary 7. *Consider the differential game for the linear model with demand intensity function (24) and parameter assumptions (25). For any initial condition t and x, and some $S \subseteq \{1, 2, \cdots, m\}$, there is a unique open-loop Nash equilibrium p^* with*

$$p_i^* = \sum_{j \in S} \frac{a_j - \frac{x_j}{t}}{2b_j} \theta_{ij} + \sum_{j \in \overline{S}} \frac{a_j}{2b_j} \theta_{ij} \,, \quad i = 1, 2, \cdots, m \,,$$

where matrix $(\theta_{ij}) = \sum_{k=0}^{\infty} {T_S}^k = (I - T_S)^{-1}$.

Proof. By Corollary 1 and the linearity of the demand intensity functions, the open-loop Nash equilibrium p^* is a solution to the system of equations

$$\begin{cases} b_i p_i^* - \sum_{j \neq i} c_{ij} p_j^* = a_i - \frac{x_i}{t}\,, & i \in S\,, \\ 2b_i p_i^* - \sum_{j \neq i} c_{ij} p_j^* = a_i\,, & i \in \overline{S}\,. \end{cases}$$

This is a linear system in p, which can be written in the matrix form as $Ap^* = d$, where $A = \Lambda(I - T_S)$ with $\Lambda = \mathrm{diag}(2b_1, \cdots, 2b_m)$, $d_i = a_i - \frac{x_i}{t}$ for $i \in S$ and $d_i = a_i$ for $i \in \overline{S}$. Since T_S is substochastic, A is invertible and $A^{-1} = (I - T_S)^{-1}\Lambda^{-1}$. Therefore, the solution is unique and $p^* = A^{-1}d = (I - T_S)^{-1}\Lambda^{-1}d = (\sum_{k=0}^{\infty} {T_S}^k)\Lambda^{-1}d$. □

Corollary 7 provides an analytic way of computing the open-loop Nash equilibrium for the linear model.

3.3 Other Log-Supermodular Models

Many demand functions other than attraction models and the linear model have the log-supermodularity property (12) and satisfy the "diagonal dominance" condition (15), for example:
the Cobb-Douglas model:

$$\lambda_i(p) = \alpha_i p_i^{-\beta_i} \prod_{j \neq i} p_j^{\beta_{ij}}\,, \text{ where } \alpha_i > 0, \beta_i > 1 \text{ for } \forall\, i \text{ and } \beta_{ij} \geq 0 \text{ for } \forall\, j \neq i\,;$$

the constant elasticity of substitution (CES) model (see Varian (1992)):

$$\lambda_i(p) = \gamma p_i^{r-1} / (\sum_{j=1}^{m} p_j^r)\,, \text{ where } r < 0 \text{ and } \gamma > 0\,;$$

the Transcendental Logarithmic (translog) model (see Christensen et al. (1973)):

$$\lambda_i(p) = \exp\left\{\alpha^i + \sum_{j=1}^{m} \beta_j^i \ln(p_j) + \sum_{j=1}^{m}\sum_{k=1}^{m} \gamma_{kj}^i \ln(p_k)\ln(p_j)\right\},$$

where $-1 < \beta_i^i < 0, \gamma_{ii}^i < 0$ for $\forall\, i$, $\beta_j^i, \gamma_{ij}^i, \gamma_{j\,i}^i \geq 0$ for $\forall\, j \neq i$, $\sum_{j=1}^m \gamma_{ji}^i < 0$ and $\sum_{j=1}^m \gamma_{ij}^i < 0$. For all these models, a unique open-loop Nash Equilibrium in the differential game is guaranteed within the specified ranges of the parameters.

In a summary, it is easier to obtain sufficient conditions of the existence and uniqueness for the general demand functions by the methodology of *supermodular games*. However, for specific demand functions with nice structures, examining the equation set of the first-order conditions might generate weaker sufficient conditions.

3.4 Robust Stability of the Open-loop Equilibrium

Condition (10) ensures supermodularity of the differential game. If demand intensity functions are twice continuously differentiable, condition (10) is the *increasing differences* property, i.e., each firm's marginal revenue of increasing its own price rises with increases in its competitor's prices, for the static game where payoff is the revenue rate function. Equilibria of supermodular games have nice stability properties. Indeed, Milgrom and Roberts (1990) proved that in supermodular games the adaptive learning algorithms converge to the set bounded by the largest and the smallest Nash equilibrium. Intuitively, a sequence of actions is consistent with adaptive learning if players "eventually abandon strategies that perform consistently badly in the sense that there exists some other strategy that performs strictly and uniformly better against every combination of what the competitors have played in the not too distant past"(Milgrom and Roberts (1990)). This includes a wide class of interesting learning dynamics, such as *Cournot tatônnement* best-response, fictitious play (see Brown (1951)), Bayesian learning (see Fudenberg and Levine (1998)) and many others. For supermodular games with a unique Nash equilibrium, we expect any adaptive learning algorithm to converge to the unique Nash equilibrium. We focus our attention only on the tatônnement scheme here since the tatônnement convergence provides an easy way to compute the equilibrium.

Due to Lemma 1, we confine the possible outcome of the tatônnement scheme in the fixed pricing policy by starting at some fixed pricing policy $p^0 = (p_1^0, p_2^0, \cdots, p_m^0) \in \mathcal{U}^{FP}$, where $\mathcal{U}^{FP}$ is the allowable fixed pricing policy set. We denote the vector of best response functions by $\Psi(p) = \big(\psi(p_{-1}), \psi(p_{-2}), \cdots, \psi(p_{-m})\big) \in \mathcal{U}^{FP}$, where $\psi(p_{-i})$ is the best response pricing strategy for any firm i in the differential game given the competitors' pricing strategies are fixed pricing policy p_{-i}. Notice that the best response problem is nothing but a deterministic version of the monopolist optimal dynamic pricing problem studied in Gallego and van Ryzin (1994, §3.1,§3.2), where the solution is a fixed pricing policy. More concretely, $\lambda_i\big(\psi(p_{-i}); p_{-i}\big) = \min\left\{\lambda_i^*(p_{-i}), n_i/t\right\}$ and thus

$$\psi(p_{-i}) = \max\left\{p_i^*(p_{-i}), p_i(n_i/t; p_{-i})\right\}, \tag{26}$$

where $p_i^*(p_{-i})$ is the maximizer of the revenue rate $r_i(p_i; p_{-i}) = p_i \lambda_i(p_i; p_{-i})$ written in terms of price and $p_i(\lambda_i; p_{-i})$ is the inverse function of demand function $\lambda_i(p_i; p_{-i})$.

Lemma 3. *If the demand intensity functions $\lambda_i(p)$, $\forall\, i$ are twice continuously differentiable and condition (10), (11) hold, $\psi(p_{-i})$ is increasing in p_{-i}.*

Proof. Since condition (10) and (11) hold, $\tilde{r}_i(p) := \log r_i(p) = \log p_i + \log \lambda_i(p)$ is concave in p_i. Then $p_i^*(p_{-i})$ satisfies the first order condition

$$\frac{1}{p_i} + \frac{1}{\lambda_i(p_i; p_{-i})} \frac{\partial \lambda_i(p_i; p_{-i})}{\partial p_i} = 0\,.$$

By the Implicit Function Theorem and condition (10),

$$\frac{\partial p_i^*(p_{-i})}{\partial p_j} = \big(- \frac{\partial \lambda_i}{\partial p_i} \frac{\partial \lambda_i}{\partial p_j} + \lambda_i \frac{\partial^2 \lambda_i}{\partial p_i \partial p_j} \big) / \big(\frac{\partial \lambda_i}{\partial p_i} \big)^2 \geq 0\,, \quad \forall\, j \neq i\,.$$

Hence, $p_i^*(p_{-i})$ is increasing in p_{-i}.

Since $\lambda_i(p_i; p_{-i})$ is increasing in p_{-i}, then $p_i(\lambda_i; p_{-i})$ is increasing in p_{-i}. Hence, $p_i(n_i/t; p_{-i})$ is increasing in p_{-i}. Therefore, by (26), $\psi(p_{-i})$ is increasing in p_{-i}. $\square$

Next we provide a direct way to verify the *increasing differences* property for the nonzero-sum differential game formulated in §2.3, which plays an important role in proving the stability of the unique equilibrium in the tatônnement scheme.

Lemma 4. *If the demand intensity functions $\lambda_i(p)$, $\forall\, i$ are twice continuously differentiable and condition (10) holds, then payoff J_i^D in the nonzero-sum differential game has increasing differences in $(p_i; p_{-i})$ for $\forall\, i$.*

Proof. Given price vector p, firm i's revenue $J_i^D(p_i; p_{-i}) = p_i \min\big(\lambda_i(p_i; p_{-i})t, n_i\big)$ in the differential game. By assumption (D2), we have $\lambda_i(p_i; p_{-i}) \geq \lambda_i(p_i; p'_{-i})$ for $p_{-i} \geq p'_{-i}$, $\forall\, i$ and thus

$$\begin{aligned} J_i^D(p_i; p_{-i}) - J_i^D(p_i; p'_{-i}) &= p_i \big[\min\big(\lambda_i(p_i; p_{-i})t, n_i\big) - \min\big(\lambda_i(p_i; p'_{-i})t, n_i\big) \big] \\ &= \begin{cases} 0\,, & \text{if } \lambda_i(p_i; p_{-i})t \geq \lambda_i(p_i; p'_{-i})t \geq n_i\,; \\ p_i[n_i - \lambda_i(p_i; p'_{-i})t]\,, & \text{if } \lambda_i(p_i; p_{-i})t \geq n_i \geq \lambda_i(p_i; p'_{-i})t\,; \\ t[r_i(p_i; p_{-i}) - r_i(p_i; p'_{-i})]\,, & \text{if } n_i \geq \lambda_i(p_i; p_{-i})t \geq \lambda_i(p_i; p'_{-i})t\,. \end{cases} \end{aligned}$$

If $\lambda_i(p_i;p_{-i})t \ge n_i \ge \lambda_i(p_i;p'_{-i})t$, by assumption (D1),

$$\frac{\partial J_i^D(p_i;p_{-i})}{\partial p_i} - \frac{\partial J_i^D(p_i;p'_{-i})}{\partial p_i} = n_i - \lambda_i(p_i;p'_{-i})t - p_i t \frac{\partial \lambda_i(p_i;p'_{-i})}{\partial p_i} \ge 0\,,$$

i.e., $J_i^D(p_i;p_{-i}) - J_i^D(p_i;p'_{-i})$ is increasing in p_i for $p_{-i} \ge p'_{-i}$.

If $n_i \ge \lambda_i(p_i;p_{-i})t \ge \lambda_i(p_i;p'_{-i})t$, condition (10) guarantees the log-supermodularity of $r_i(p_i;p_{-i}) = p_i\lambda_i(p_i;p_{-i})$ in $(p_i;p_{-i})$, thus ensures *increasing differences* of $\log\big(r_i(p_i;p_{-i})\big)$ in $(p_i;p_{-i})$ and then for any $p_{-i} \ge p'_{-i}$,

$$\frac{\partial r_i(p_i;p_{-i})}{\partial p_i} - \frac{\partial r_i(p_i;p'_{-i})}{\partial p_i} \ge r_i(p)\left(\frac{\partial \log r_i(p_i;p_{-i})}{\partial p_i} - \frac{\partial \log r_i(p_i;p'_{-i})}{\partial p_i}\right) \ge 0\,.$$

Therefore, we always have $J_i^D(p_i;p_{-i}) - J_i^D(p_i;p'_{-i})$ is increasing in p_i for any $p_{-i} \ge p'_{-i}$. □

Lemma 4 essentially provides another way to argue the supermodularity of the differential game formulated in §2.3 under the existence sufficient condition (10) since under the assumption of differentiability the increasing difference property is equivalent to supermodularity. Now we are ready to present results and examples on the stability of the open-loop Nash equilibrium. Let us define $P^+ = \bigcap_{i=1}^m \{p \in \mathcal{U}^{FP} : p_i \ge \psi(p_{-i})\}$, $P^- = \bigcap_{i=1}^m \{p \in \mathcal{U}^{FP} : p_i \le \psi(p_{-i})\}$ and $P = \{p \in \mathcal{U}^{FP} : \exists\, \underline{p}^0 \in P^-, \overline{p}^0 \in P^+,\ \text{s.t. } \underline{p}^0 \le p \le \overline{p}^0\}$.

Theorem 4. *If the demand intensity functions $\lambda_i(p)$, $\forall\, i$ are twice continuously differentiable and condition (10), (11) hold, then a tatônnement best-response scheme starting at any fixed pricing policy $p^0 \in P$ converges monotonically downwards (upwards) to the unique equilibrium p^* in the nonzero-sum differential game.*

Proof. By Lemma 4, the payoff of firm i, $\forall\, i$ in the differential game has *increasing differences* in $(p_i;p_{-i})$. By Lemma 3, $\psi(p_{-i})$ is increasing in p_{-i} for any i. The uniqueness of $\lambda_i^*(p_{-i})$ and $\psi(p_{-i})$ is guaranteed by condition (11) since it ensures the strict log-concavity of $r_i(p_i;p_{-i})$ in p_i, and thus ensures the strict log-concavity of $r_i(\lambda_i;p_{-i})$ in λ_i.

By Vives (1990, Theorem 5.1), a tatônnement scheme starting at any $p^0 \in P^+$ (or P^-) converges monotonically downwards (or upwards) to the unique equilibrium p^*, where $\Psi(p^*) = p^*$, i.e., $\psi(p^*_{-i}) = p^*_i$ for $\forall\, i$. With any starting fixed pricing policy $p^0 \in P$, we define $p^{k+1} = \Psi(p^k)$,

$\underline{p}^{k+1} = \Psi(\underline{p}^k)$ and $\overline{p}^{k+1} = \Psi(\overline{p}^k)$ for any $k \geq 0$. By the monotonicity of $\Psi(\cdot)$, we have $\underline{p}^k \leq p^k \leq \overline{p}^k$. Since $\lim_{k\to\infty} \underline{p}^k = p^*$ and $\lim_{k\to\infty} \overline{p}^k = p^*$, we have $\lim_{k\to\infty} p^k = p^*$. □

Corollary 8. *For differential games arising from the attraction demand intensity function (16) and the linear demand intensity function (24) with unique open-loop Nash equilibrium p^*, a tatônnement best-response scheme starting at any fixed pricing policy $p^0 \in \mathcal{U}^{FP}$ converges linearly to p^*.*

Proof. The result can be easily adapted from the proofs of Theorem 2 and 3 in Gallego et al. (2005a). □

Corollary 9. *If the demand intensity functions $\lambda_i(p)$, $\forall\, i$ are twice continuously differentiable and condition (10), (11) hold, $p^*(t, x)$ is decreasing in x/t.*

Proof. Without loss of generality, we assume $x_i^1/t_1 < x_i^2/t_2$ for some i and $x_j^1/t_1 = x_j^2/t_2$ for $\forall\, j \neq i$. Starting from the equilibrium price vector $p^*(t_1, x^1)$, we use tatônnement best-response scheme to compute the equilibrium price vector $p^*(t_2, x^2)$. Since $p_i(n_i/t; p_{-i})$ is decreasing in n_i/t in the best-response function $\psi(p_{-i})$, by Lemma 3, the tatônnement scheme would converge monotonically downwards and end up at a smaller equilibrium price vector. □

Having the tatônnement scheme as a computational tool for the open-loop Nash equilibrium, we list here several numerical examples.

Example 2. We consider competition among $m = 5$ firms and use a tatônnement scheme for the MNL model with attraction functions $a_i(p_i) = \beta_i \exp\{-\alpha_i p_i\}$, $i = 1, 2, \cdots, 5$. Table 2 shows the equilibrium price vector p^* and demand vector λ^* for the MNL model, provided no-purchase value $a_0 = 0.25$, $(\alpha_1, \alpha_2, \cdots, \alpha_5) = (.5, .75, 1, 1.25, 1.5)$, $(\beta_1, \beta_2, \cdots, \beta_5) = (.5, .75, 1, 1.25, 1.5)$ under different initial conditions, from all the firms applying the revenue rates to all the firms applying the run-out rates. Notice that for this example when x_i/t decreases for some i while the corresponding ratio is kept fixed for players, the whole vector of optimal prices p^* increases. Intuitively, if x_i/t decreases, firm i tends to price higher and other firms in the market react by increasing their prices accordingly to keep things at an equilibrium.

Figure 2 illustrates the linear convergence of tatônnement scheme for the MNL model with different no-purchase value a_0, provided $(\alpha_1, \alpha_2, \cdots, \alpha_5) = (.5, .75, 1, 1.25, 1.5)$, $(\beta_1, \beta_2, \cdots, \beta_5) = (.5, .75, 1, 1.25, 1.5)$, initial condition $\left(\frac{x_1}{t}, \frac{x_2}{t}, \cdots, \frac{x_5}{t}\right) = (0.09, 0.5, 0.17, 0.5, 0.24)$ and starting point

$\frac{x}{t} = \left(\frac{x_1}{t}, \frac{x_2}{t}, \cdots, \frac{x_5}{t}\right)$	$p^* = (p_1^*, p_2^*, \cdots, p_5^*)$	$\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_5^*)$
(.50,.50,.50,.50,.50)	(2.2144,1.5464,1.2114,1.0093,.8737)	(.10,.14,.17,.21,.24)
(.09,.50,.50,.50,.50)	(2.3793,1.5484,1.2133,1.0112,.8756)	(.09,.14,.18,.21,.24)
(.09,.13,.50,.50,.50)	(2.4081,1.6557,1.2163,1.0141,.8784)	(.09,.13,.18,.21,.24)
(.09,.13,.17,.50,.50)	(2.4407,1.6775,1.2775,1.0174,.8816)	(.09,.13,.17,.21,.24)
(.09,.13,.17,.21,.50)	(2.4636,1.6927,1.2890,1.0406,.8839)	(.09,.13,.17,.21,.25)
(.09,.13,.17,.21,.24)	(2.5370,1.7417,1.3257,1.0700,.9242)	(.09,.13,.17,.21,.24)

Table 2: Open-loop Equilibrium for MNL with Different Initial Conditions

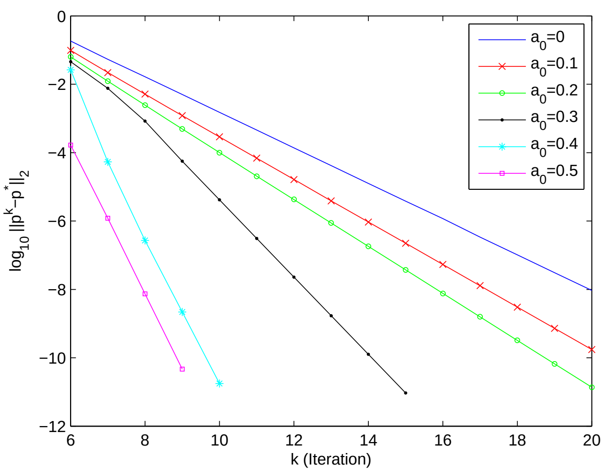

Figure 2: Linear Convergence of Tatônnement for MNL with Different No-purchase Values

$(p_1^0, p_2^0, \cdots, p_5^0) = (4, 3, 3, 3, 3)$. We see that with larger no-purchase value, the linear convergence is faster.

Example 3. Following the competitive setting of Example 2 we numerically examine the impact on revenues if some (rational) firms use their equilibrium strategies while others (irrational) firms deviate from their equilibrium strategies. We use $a_0 = 0.25$, $(\alpha_1, \alpha_2, \cdots, \alpha_5) = (.5, .75, 1, 1.25, 1.5)$ and $(\beta_1, \beta_2, \cdots, \beta_5) = (.5, .75, 1, 1.25, 1.5)$ in the MNL model. Table 3 specifies combinations of irrational firms who ignore the inventory effect and always try to maximize their revenue rates. There is a severe loss for any irrational firm, compared to the revenue it could make by using dynamic pricing. The extent of the loss is roughly proportional to the inventory level: with less inventory, the firm suffers more by pricing irrationally. When irrational firms with limited inventories keep their prices low, they have two effects on their competitors: 1. before irrational firms run out of stock, their relatively low prices hurt competitors with more price-sensitive competitors taking larger losses; 2. after irrational firms run out of stock, competitors can increase prices and improve profits. Price increase after some firm i runs out of inventory can be justified by Lemma 3 since best-response prices $\psi(p_{-i})$ of rational firms and revenue-rate maximizing prices $p_i^*(p_{-i})$ of irrational firms are all increasing functions in p_{-i} and firm i adjusts price to ∞ after inventory turns zero. The sum of these two contrary effects depends on which one is more significant. Low revenue-rate maximizing prices of irrational firms can mislead their competitors to think they have abundant inventories. In spite of effect 2, rational firms with limited inventories suffer more than rational firms with abundant inventories because when they increase prices after some irrational firms run out of inventory, they have already consumed parts of their limited inventories at a relatively low price due to the misleading low prices of irrational firms.

Table 4 shows that there is a loss for both irrational firms and their rational competitors when irrational firms use market clearing price when they have abundant inventories. In Table 5, we consider a situation where an irrational firm selects a price randomly between the revenue-rate maximizing price and the market clearing price. The Table suggests that using such a policy results in loss for the irrational firms and may cause gains to strategic competitors using dynamic pricing. We use initial state $(x_1/t, x_2/t, \cdots, x_5/t) = (.20, .20, .17, .20, .23)$ for the Monte-Carlo experiment presented in Table 5, where firm 3, 4 and 5 have limited inventories and again consider effect 2 in the simulation when some irrational firms run out of inventory.

4 Fixed-pricing Heuristic as ϵ Nash equilibrium

The nonzero-sum differential game in §2.3 suggests a simple fixed pricing (FP) heuristic for the original nonzero-sum stochastic game formulated in §2.2. In this section, we show such a tractable heuristic has good asymptotic properties.

For any $\epsilon \geq 0$, an ϵ Nash equilibrium in a stochastic game is a set of pricing policies, one for each firm, such that no firm can gain more than ϵ by choosing a different strategy, given that all the other players do not change their strategies. More precisely, for any $\epsilon \geq 0$, u^* is called an ϵ *Nash equilibrium* if

$$J_i\big(t, n_i, u_i, u_{-i}^*\big) \leq J_i\big(t, n_i, u_i^*, u_{-i}^*\big) + \epsilon\,,$$

Irrational Set	$\frac{x}{t} = \left(\frac{x_1}{t}, \frac{x_2}{t}, \cdots, \frac{x_5}{t}\right)$	$\frac{J^D}{t} = \left(\frac{J_1^D}{t}, \frac{J_2^D}{t}, \cdots, \frac{J_5^D}{t}\right)$	Run-out Set
$\emptyset$	(.05,.50,.50,.50,.50)	(.1831,.2267,.2246,.2223,.2196)	$\{1\}$
$\{1\}$	(.05,.50,.50,.50,.50)	(60%,100%,100%,100%,100%)	$\emptyset$
$\emptyset$	(.05,.06,.50,.50,.50)	(.1961,.1748,.2544,.2510,.2474)	$\{1,2\}$
$\{1\}$	(.05,.06,.50,.50,.50)	(57%,100%,100%,100%,100%)	$\{2\}$
$\{2\}$	(.05,.06,.50,.50,.50)	(100%,54%,100%,100%,100%)	$\{1\}$
$\{1,2\}$	(.05,.06,.50,.50,.50)	(53%,53%,101%,101%,101%)	$\emptyset$
$\emptyset$	(.05,.06,.08,.50,.50)	(.2181,.1924,.1924,.3072,.3011)	$\{1,2,3\}$
$\{1\}$	(.05,.06,.08,.50,.50)	(52%,100%,100%,100%,100%)	$\{2,3\}$
$\{2\}$	(.05,.06,.08,.50,.50)	(100%,50%,100%,100%,100%)	$\{1,3\}$
$\{3\}$	(.05,.06,.08,.50,.50)	(100%,100%,52%,100%,100%)	$\{1,2\}$
$\{1,2\}$	(.05,.06,.08,.50,.50)	(49%,49%,81%,104%,104%)	$\{3\}$
$\{1,3\}$	(.05,.06,.08,.50,.50)	(49%,80%,51%,103%,103%)	$\{2\}$
$\{2,3\}$	(.05,.06,.08,.50,.50)	(81%,49%,50%,103%,103%)	$\{1\}$
$\{1,2,3\}$	(.05,.06,.08,.50,.50)	(47%,48%,49%,102%,102%)	$\emptyset$

Table 3: Effects on Payoffs if Irrational Firms Maximize their Revenue Rates

Irrational Set	$\frac{x}{t} = \left(\frac{x_1}{t}, \frac{x_2}{t}, \cdots, \frac{x_5}{t}\right)$	$\frac{J^D}{t} = \left(\frac{J_1^D}{t}, \frac{J_2^D}{t}, \cdots, \frac{J_5^D}{t}\right)$	Run-out Set
$\{\emptyset\}$	(.11,.15,.19,.22,.25)	(.2144,.2131,.2114,.2093,.2071)	$\{\emptyset\}$
$\{1\}$	(.11,.15,.19,.22,.25)	(99%,98%,98%,98%,98%)	$\{1\}$
$\{1,2\}$	(.11,.15,.19,.22,.25)	(96%,97%,96%,96%,96%)	$\{1,2\}$
$\{1,2,3\}$	(.11,.15,.19,.22,.25)	(91%,93%,93%,92%,92%)	$\{1,2,3\}$
$\{1,2,3,4\}$	(.11,.15,.19,.22,.25)	(82%,84%,85%,86%,84%)	$\{1,2,3,4\}$
$\{1,2,3,4,5\}$	(.11,.15,.19,.22,.25)	(39%,44%,47%,50%,53%)	$\{1,2,3,4,5\}$

Table 4: Effects on Payoffs if Irrational Firms Use Market Clearing Prices

for any $\forall\, i = 1, 2, \cdots, m$ and $(u_i, u^*_{-i}) = \left(u^*_1, \cdots, u^*_{i-1}, u_i, u^*_{i+1}, \cdots, u^*_m\right) \in \mathcal{U}$.

When $m = 1$, there is no competition but monopoly. Let us review an important bound result in the monopoly pricing problem then proceed to present our main result of this section. In the monopolistic setting, given a pricing policy $u \in \mathcal{U}$, an initial stock $n > 0$ and a sales horizon $t > 0$, we denote the expected revenue by $J_u(t, n) := \mathbb{E}_u\left[\int_0^t p_s\, dN_s\right]$, where $J_u(0, n) = 0$ for $\forall\, n$ and $J_u(t, 0) = 0$ for $\forall\, t$. The optimal total expected revenue generated over $[0, t]$ is denoted by $J^*(t, n)$, i.e., $J^*(t, n) := \sup_{u \in \mathcal{U}} J_u(t, n)$. The maximal total revenue generated over $[0, t]$ in the deterministic fluid problem is denoted by $J^D(t, n)$. Let $J^{FP}(t, n)$ denote the total expected revenue for the stochastic problem under the fixed pricing heuristic suggested by the optimal solution to the fluid problem (see Gallego and van Ryzin (1994, §2.2,§3.1,§3.3)).

Irrational Set	Distributions of Price	$\mathbb{E}(\frac{J^D}{t}) = (\frac{\mathbb{E}(J_1^D)}{t}, \frac{\mathbb{E}(J_2^D)}{t}, \cdots, \frac{\mathbb{E}(J_5^D)}{t})$
$\emptyset$	N/A	(.2259,.2244,.2194,.2162,.2137)
$\{4\}$	$p_4 \sim U(.8, 1.5)$	(120%,109%,96%,88%,96%)
$\{5\}$	$p_5 \sim U(.8, 1.5)$	(121%,113%,103%,102%,93%)

Table 5: Effects on Payoffs if Irrational Firms Use Randomly Selected Prices

Lemma 5. *Consider the monopoly optimal dynamic pricing problem.*

$$J^*(t,n)\big(1 - \frac{1}{2\sqrt{\min\{n, \lambda^* t\}}}\big) \le J^{FP}(t,n) \le J^*(t,n) \le J^D(t,n)\,. \tag{27}$$

Proof. Combining Theorem 2 and 3 in Gallego and van Ryzin (1994), we have the result. □

For the nonzero-sum stochastic game, the problem for firm i to maximize its own expected total revenue while all the other firms take a fixed pricing policy is nothing but a monopoly optimal dynamic pricing problem since the demand intensity function of firm i is time invariant. Next we have asymptotic optimality of the fixed pricing policy in the monopolistic problem extended to ϵ Nash equilibrium of the fixed pricing policy in the multi-player game.

Theorem 5. *Consider the nonzero-sum stochastic game. For any $\epsilon > 0$, the fixed pricing policy $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$ for some $S \subseteq \{1, 2, \cdots, m\}$ given by (4)-(5) is an ϵ Nash equilibrium for the initial condition*

$$\begin{cases} t\big(\frac{p_i^*\lambda_i(p^*)}{2\epsilon}\big)^2 \le \frac{n_i}{t} < \lambda_i^*(p_{-i}^*)\,, & i \in S\,, \\ t\big(\frac{p_i^*\lambda_i(p^*)}{2\epsilon}\big)^2 \le \lambda_i^*(p_{-i}^*) \le \frac{n_i}{t}\,, & i \in \overline{S}\,, \end{cases} \tag{28}$$

where $\lambda_i^*(p_{-i})$ is the maximizer of function $r_i(\lambda_i; p_{-i})$ with respect to λ_i.

Proof. With $p_{-i}^* = \big(p_1^*, p_2^*, \cdots, p_{i-1}^*, p_{i+1}^*, \cdots, p_m^*\big)$ fixed, in the stochastic game any firm i just faces the monopoly stochastic dynamic pricing problem discussed in Gallego and van Ryzin (1994). Since p^* is the open-loop Nash equilibrium of the differential game, p_i^* is the fixed pricing heuristic suggested by the monopoly deterministic pricing problem. By Lemma 5, we have

$$J_i^*(t, n_i, p_{-i}^*)\left(1 - \frac{1}{2\sqrt{\min\{n_i, \lambda_i^*(p_{-i}^*)t\}}}\right) \le J_i(t, n_i, p_i^*, p_{-i}^*) \le J_i^*(t, n_i, p_{-i}^*) \le J_i^{D,*}\,, \tag{29}$$

where

$$J_i^*(t, n_i, p_{-i}^*) = \sup_{(p_i, p_{-i}^*) \in \mathcal{U}} J_i(t, n_i, p_i, p_{-i}^*)\,, \tag{30}$$

and $J_i^{D,*}$ is the value function for firm i in the differential game under the fixed pricing policy p^*.

For any $\epsilon > 0$, if the initial condition satisfies

$$\frac{1}{2\sqrt{\min\{n_i, \lambda_i^*(p_{-i}^*)t\}}} \leq \frac{\epsilon}{J_i^{D,*}}\,, \quad \text{for } i = 1, 2, \cdots, m\,, \tag{31}$$

then by (29),

$$\begin{aligned} J_i(t, n_i, p_i^*, p_{-i}^*) &\geq J_i^*(t, n_i, p_{-i}^*)\left(1 - \frac{1}{2\sqrt{\min\{n_i, \lambda_i^*(p_{-i}^*)t\}}}\right) \\ &\geq J_i^*(t, n_i, p_{-i}^*)\left(1 - \frac{\epsilon}{J_i^{D,*}}\right) \\ &\geq J_i^*(t, n_i, p_{-i}^*)\left(1 - \frac{\epsilon}{J_i^*(t, n_i, p_{-i}^*)}\right) \\ &\geq J_i^*(t, n_i, p_{-i}^*) - \epsilon\,. \end{aligned}$$

Thus, by definition (30),

$$J_i(t, n_i, p_i, p_{-i}^*) \leq J_i(t, n_i, p_i^*, p_{-i}^*) + \epsilon\,, \quad \forall\, i = 1, 2, \cdots, m\,, \tag{32}$$

for any $(p_i, p_{-i}^*) \in \mathcal{U}$.

Since p^* is the open-loop Nash equilibrium under initial condition

$$\begin{cases} \frac{n_i}{t} < \lambda_i^*(p_{-i}^*)\,, & i \in S\,, \\ \frac{n_i}{t} \geq \lambda_i^*(p_{-i}^*)\,, & i \in \overline{S}\,, \end{cases} \tag{33}$$

then condition (31) becomes

$$\begin{cases} \frac{1}{2\sqrt{n_i}} \leq \frac{\epsilon}{p_i^* \lambda_i(p^*)t}\,, & i \in S\,, \\ \frac{1}{2\sqrt{\lambda_i^*(p_{-i}^*)t}} \leq \frac{\epsilon}{p_i^* \lambda_i(p^*)t}\,, & i \in \overline{S}\,, \end{cases} \tag{34}$$

since $J_i^{D,*} = p_i^* \lambda_i(p^*)t$ for $\forall\, i$. Combining (33) and (34), we have the initial condition (28) under which p^* is an ϵ Nash equilibrium. $\square$

Theorem 5 implies that the fixed pricing heuristic is an asymptotic Nash equilibrium ($\epsilon \to 0$) in only one limiting case: the planning horizon t is very short ($t \ll 1$) and there are very few items n_i to sell for $\forall\, i \in S$ ($n_i \ll 1, \forall\, i \in S$). The reason is that when $n, t \gg 1$, the distance between Nash equilibrium and the heuristic is large in absolute value since $J_i^{D,*} \gg 1$ although it is relatively small with respect to $J_i^{D,*}$. Then we have the necessity to definite *relative ϵ Nash equilibrium* to accommodate other limiting cases. We call u^* a *relative ϵ Nash equilibrium* if

$$\frac{J_i(t, n_i, u_i, u_{-i}^*)}{J_i(t, n_i, u_i^*, u_{-i}^*)} \le 1 + \epsilon\,,$$

for any $\forall\, i = 1, 2, \cdots, m$ and $(u_i, u_{-i}^*) = (u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_m^*) \in \mathcal{U}$.

Theorem 6. *Consider the nonzero-sum stochastic game. For any $\epsilon > 0$, the fixed pricing policy $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$ for some $S \subseteq \{1, 2, \cdots, m\}$ given by (4)-(5) is a relative ϵ Nash equilibrium for the initial condition*

$$\begin{cases} \left(\frac{1}{2\epsilon}\right)^2 \le n_i < \lambda_i^*(p_{-i}^*)t\,, & i \in S\,, \\ \left(\frac{1}{2\epsilon}\right)^2 \le \lambda_i^*(p_{-i}^*)t \le n_i\,, & i \in \overline{S}\,, \end{cases}$$

where $\lambda_i^*(p_{-i})$ is the maximizer of function $r_i(\lambda_i; p_{-i})$ with respect to λ_i.

Proof. Replace ϵ in Theorem 5 by $J_i^{D,*}\epsilon$. $\square$

Theorem 6 shows that the fixed pricing heuristic is an asymptotic relative Nash equilibrium ($\epsilon \to 0$) in the following limiting case: for those firms using the run-out rate to price, the number of items in stock is large ($n_i \gg 1$) and there is plenty of time to sell them ($n_i < \lambda_i^*(p_{-i}^*)t$); for those firms using the revenue rate to price, there are enough items in stock to sustain the potential demand ($n_i \ge \lambda_i^*(p_{-i}^*)t$) at the revenue maximizing price ($\lambda_i^*(p_{-i}^*)t \gg 1$). Thus, we see that if the *volume* of expected sales in the game is large, the heuristic performs quite close to the relative Nash equilibrium.

Example 4. We revisit the competition scenario among $m = 5$ firms in Example 2 where $a_0 = 0.25$, $(\alpha_1, \alpha_2, \cdots, \alpha_5) = (.5, .75, 1, 1.25, 1.5)$, $(\beta_1, \beta_2, \cdots, \beta_5) = (.5, .75, 1, 1.25, 1.5)$ are the parameters in the MNL attraction model. In Table 6, we numerically demonstrate the asymptotic optimality of the fixed pricing heuristic suggested by the open-loop Nash equilibrium. When $t =$

$t = 100$	$p^{FP}(t, n)$	$J^{FP}(t, n)/J^D(t, n)$
$n = 1e$	(10.4940,7.5367,5.9402,4.9307,4.2304)	(63.22%,63.41%,63.20%,63.34%,63.17%)
$n = 2e$	(8.9996,6.5404,5.1930,4.3329,3.7323)	(72.89%,72.68%,72.93%,73.12%,72.96%)
$n = 3e$	(8.0744,5.9235,4.7303,3.9628,3.4239)	(77.55%,77.67%,77.71%,77.63%,77.64%)
$n = 4e$	(7.3778,5.4591,4.3820,3.6841,3.1917)	(80.38%,80.52%,80.64%,80.37%,80.57%)
$n = 5e$	(6.8024,5.0755,4.0943,3.4540,2.9999)	(82.30%,82.57%,82.56%,82.38%,82.40%)
$n = 6e$	(6.2998,4.7405,3.8430,3.2529,2.8323)	(83.98%,83.93%,83.98%,84.08%,83.92%)
$n = 7e$	(5.8432,4.4361,3.6148,3.0703,2.6802)	(85.08%,85.08%,84.98%,85.12%,85.20%)
$n = 8e$	(5.4161,4.1513,3.4012,2.8995,2.5378)	(86.02%,85.93%,85.98%,85.94%,86.15%)
$n = 9e$	(5.0065,3.8783,3.1964,2.7356,2.4012)	(86.78%,86.87%,86.80%,86.76%,86.82%)
$n = 10e$	(4.6052,3.6107,2.9957,2.5751,2.2675)	(87.51%,87.51%,87.48%,87.39%,87.59%)
$n = 11e$	(4.2039,3.3432,2.7951,2.4146,2.1337)	(88.10%,88.00%,88.04%,88.02%,88.04%)
$n = 12e$	(3.7943,3.0701,2.5903,2.2507,1.9972)	(88.50%,88.63%,88.63%,88.65%,88.62%)
$n = 13e$	(3.3671,2.7854,2.3767,2.0799,1.8548)	(88.96%,89.01%,89.05%,89.03%,89.07%)
$n = 14e$	(2.9106,2.4810,2.1485,1.8973,1.7026)	(89.49%,89.42%,89.41%,89.47%,89.41%)
$n = 15e$	(2.4080,2.1460,1.8971,1.6962,1.5351)	(89.69%,89.85%,89.77%,89.73%,89.81%)
$n = 16e$	(2.3219,1.8976,1.7109,1.5472,1.4109)	(95.11%,90.06%,90.15%,89.99%,90.07%)
$n = 20e$	(2.2339,1.5655,1.2300,1.1088,1.0456)	(99.76%,98.72%,93.74%,91.19%,91.14%)
$n = 25e$	(2.2144,1.5464,1.2114,1.0093,0.8737)	(99.96%,99.90%,99.46%,97.75%,94.20%)
$n = 30e$	(2.2144,1.5464,1.2114,1.0093,0.8737)	(99.99%,100%,100%,99.76%,98.96%)
$n = 37e$	(2.2144,1.5464,1.2114,1.0093,0.8737)	(100%, 100%, 100%, 100%, 100%)

Table 6: Asymptotic Optimality of the Fixed Pricing Heuristic, where $e = (1, 1, 1, 1, 1)$

100, $(n_1, n_2, \cdots, n_5) > 30e$ the relative performance of the FP heuristic[7] is within 1% of the deterministic upper bound, thus within 1% of the stochastic Nash equilibrium. We notice that less price-sensitive firm becomes nearly optimal for smaller initial inventory: when $(n_1, n_2, \cdots, n_5) > 17e$, the relative performance of the FP heuristic for firm $i = 1$ is within 1% of its value at the stochastic Nash equilibrium. These results suggest that even for moderate sized problems, the fixed pricing heuristic performs quite well. They also suggest that if demand functions are well-known and prices can be set freely, then one should not see great benefits from the highly dynamic pricing practices for moderate to large sized problems.

5 Repeated Games

In this section, we consider a repetition of the single-stage multi-player nonzero-sum noncooperative games formulated in §2.2 and §2.3. The model of a repeated game has two versions: finitely repeated

[7] The performance $J^{FP}(t, n)$ of the fixed pricing heuristic in the stochastic game is obtained by simulation from 100,000 sample paths.

game and infinitely repeated game. For the finitely repeated game, we state the following conclusion that is an immediate result from *repeated game* theory.

Lemma 6. *If the single-stage nonzero-sum differential game has a unique open-loop Nash equilibrium* $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$*, then* p^* *conducted in every stage is the unique open-loop subgame-perfect equilibrium in any finitely repeated differential game.*

The result does not hold when the single stage game has more than one Nash equilibrium (see Benoît and Krishna (1985)).

Theorem 7. *Consider a* K*-time repetition of the nonzero-sum stochastic game. For any* $\epsilon > 0$*, if the corresponding single-stage differential game has a unique open-loop Nash equilibrium* $p^* = (p_1^*, p_2^*, \cdots, p_m^*)$ *under the initial condition*

$$\begin{cases} t\big(\frac{p_i^* \lambda_i(p^*)K}{2\epsilon}\big)^2 \le \frac{n_i}{t} < \lambda_i^*(p_{-i}^*)\,, & i \in S\,, \\ t\big(\frac{p_i^* \lambda_i(p^*)K}{2\epsilon}\big)^2 \le \lambda_i^*(p_{-i}^*) \le \frac{n_i}{t}\,, & i \in \overline{S}\,, \end{cases}$$

for some $S \subseteq \{1, 2, \cdots, m\}$*, then the fixed pricing policy* p^* *conducted in every stage sustains a subgame-perfect* ϵ *equilibrium for the finitely repeated stochastic game.*

Proof. By Theorem 5, p^* is an ϵ/K Nash equilibrium for the single stage stochastic game. Then the result follows from Lemma 6 and the definition of a subgame-perfect ϵ equilibrium. ☐

For the infinitely repeated game, we have a similar situation as the *Prisoner's Dilemma* (see Osborne and Rubinstein (1994, Chap.8)), where some strategy in the joint strategy space is not a Nash equilibrium while strictly dominating the payoff of the Nash equilibrium for each player. For example, let us consider a two-player differential game with linear demand functions: $\lambda_1 = 2 - (p_1 - 0.5p_2)$ and $\lambda_2 = 2 - (p_2 - 0.5p_1)$. For the simple case with enough initial stocks to run at the revenue rates for both players, i.e., $n_1/t \gg 1$ and $n_2/t \gg 1$, we have the fixed pricing open-loop Nash equilibrium $(p_1^*, p_2^*) = (4/3, 4/3)$ with payoff $(r_1^*, r_2^*) = (16t/9, 16t/9)$. However, there are numerous points in the strategy space whose payoffs for both firms beat the payoff at the Nash equilibrium, e.g., fixed pricing policy $(p_1, p_2) = (2, 2)$ with payoff $(r_1, r_2) = (2t, 2t)$.

Similar to the *Prisoner's Dilemma*, there exist stable strategies to sustain mutually desirable payoffs in the infinitely repeated differential game. The basic idea behind is that if the game is played repeatedly then the mutually desirable outcome is stable if each firm believes that a deviation will terminate the cooperation, resulting in a subsequent loss by getting "punishment" from other firms. There are a series of "folk theorems" in the *game theory* literature dealing with "cooperation" in the infinitely repeated game. We have our version of "folk theorem" here for the nonzero-sum differential game:

Proposition 4. *Consider the infinitely repeated nonzero-sum differential game with average discounted payoff $\Lambda_i(\delta) = (1-\delta)\sum_{k=0}^{\infty}\delta^k r_i^k$ for firm i, $\forall\, i$, where δ is the discount factor and r_i^k is firm i's payoff in the k-th stage-game. Suppose R^* is the set of payoff vectors that strictly Pareto dominate the maximal Nash equilibrium payoff vector r^* in the single-stage game. Then for $\forall\, r \in R^*$, $\exists\, \underline{\delta} \in (0,1)$ such that $\forall\, \delta \in [\underline{\delta}, 1)$, r can be sustained as the payoff of a subgame-perfect equilibrium in the infinitely repeated game.*

Proof. The result is adapted from Friedman (1971) and we use the payoff r^* of the single-stage game's open-loop Nash equilibrium to punish any deviator. □

In applying the model of a repeated game to specific situations, we need to determine whether a finite or infinite horizon is appropriate. A model should try to capture what the players perceive. For the situation that the players clearly believe there is an end to the game, we should model it as a finitely repeated game and the result of Theorem 7 holds. For the situation where a stage-game is played so frequently that the players always perceive there is a next round or a positive probability of playing the stage game again, the strategic interaction is better to be modeled as an infinitely repeated game and the result of Proposition 4 holds.

For the infinitely repeated game, "folk theorem"-like results state that any dominant payoff vector to the equilibrium in the stage-game can be sustained as the payoff of a subgame-perfect equilibrium in the infinitely repeated game. However, which specific dominant payoff vector is sustained as a result depends on how the firms "cooperate" with each other, i.e., how they react to the "threats" imposed by others and how they learn to play in the repetition. This result seems to be vague in that which equilibrium eventually is settled upon is determined by each firm's strategic behavior that is hard to model. Nevertheless, the open-loop Nash equilibrium in the single-stage game still provides a lower bound to how well each firm can perform in the infinitely repeated differential game.

6 Extensions to the Basic Problem

In this section, we examine several extensions to the basic single-stage game formulated in §2.2 and §2.3.

6.1 Demand is a Function of Price and Quality

Suppose at time zero, firm $i = 1, 2, \cdots, m$ has inventory $n_i^q \in \mathbb{Z}_+$ units of the same product of class $q = 1, 2, \cdots, l$ and a finite time $t > 0$ to sell them. We assume that classes are indexed such that lower index has higher quality and that the fare for a class with lower index will be priced higher than the fare for a class with higher index, i.e., $p_i^1(s) \geq p_i^2(s) \geq \cdots \geq p_i^l(s)$ at any $s \in [0, t]$ for any i. Now the demand intensity depends not only on the price vector but also on the quality or service level of the item. Let $\lambda_i^q(s) = \lambda_i^q\big(p^q(s)\big)$[8] denote the *regular* demand intensity function for class $q = 1, 2, \cdots, l$ of firm $i = 1, 2, \cdots, m$, where $p^q(s) = \big(p_1^q(s), p_2^q(s), \cdots, p_m^q(s)\big) \in \mathcal{U}^q$ for $\forall\, q$ is the price vector of quality q across the industry under the competition and $\mathcal{U}^q$ is the joint allowable Markovian pricing policy space for quality q, $\forall\, q$. At the same price level, we assume that the demand intensity of a higher quality class is larger than the demand intensity of a lower quality class, i.e., $\lambda_i^1(p) \geq \lambda_i^2(p) \geq \cdots \geq \lambda_i^l(p)$ for any i. Thus, the revenue rate as a function of price vector for a higher quality class is larger than the revenue rate for a lower quality class, i.e., $r_i^1(p) \geq r_i^2(p) \geq \cdots \geq r_i^l(p)$ for any i. The goal of each firm is to price every class dynamically to maximize the total revenue combining all classes over the planning horizon. We assume a higher quality class can be downgraded and consequently priced at the fare of a lower quality class if there expected to be an overstock of the higher quality class.

As demonstrated in the basic problem, it is very hard to solve the Markovian (close-loop) Nash equilibrium of the nonzero-sum stochastic game. Instead, we formulate the corresponding nonzero-sum differential game as follows: given pricing policy $p^q \in \mathcal{U}^q$ for $\forall\, q = 1, 2, \cdots, l$, initial stock $x_i^q > 0$, $i = 1, 2, \cdots, m$, $q = 1, 2, \cdots, l$ and a sales horizon $t > 0$, we denote the total profit for firm i by

$$J_i^Q := \sum_{q=1}^{l} \int_0^t p_i^q \lambda_i^q\big(p^q(s)\big)\, ds = \sum_{q=1}^{l} \int_0^t r_i^q\big(p^q(s)\big)\, ds\,, \tag{35}$$

[8] In the next section, we consider a generalized case to the scenario when the demand for a specific quality or fare class depends on prices of all fare classes across the industry.

and the restrictions on the inventory availability are

$$\int_0^t \lambda_i^q\big(p^q(s)\big)\,ds \le \sum_{k=q}^{l} x_i^k\,, \quad \forall\, q = 1, 2, \cdots, l\,. \tag{36}$$

We construct the following algorithm to compute the open-loop Nash equilibrium of problem (35)-(36). We call $T\big(t, x, \lambda(p)\big)$ the procedure of the tatônnement best-response scheme presented in §3.4 to find the open-loop Nash equilibrium p^* in the basic nonzero-sum differential game with inventory $x = (x_1, x_2, \cdots, x_m)$, demand intensity function $\lambda(p) = \big(\lambda_1(p), \lambda_2(p), \cdots, \lambda_m(p)\big)$ and a planning horizon t.

Algorithm 1. $q := 1$;

while $q <= l$

apply $T\big(t, x^q, \lambda^q(p^q)\big)$ to find $p^{*,q}$;

$x^{q+1} \leftarrow x^{q+1} + (x^q - \lambda^{*,q} t)$;

$q \leftarrow q + 1$;

end

Theorem 8. *If demand intensity functions $\lambda^q(p)$ for $\forall\, q$ are twice continuously differentiable and conditions (10), (11) hold for $\forall\, q$, Algorithm 1 solves for the unique open-loop Nash equilibrium of the differential game (35)-(36), which is a (relative) ϵ Nash equilibrium for the stochastic game asymptotically.*

Proof. By the same argument of Hamiltonians as in Lemma 1, the open-loop Nash equilibrium of the nonzero-sum differential game (35)-(36) is a fixed pricing policy $p^{*,q}(s) = \big(p_1^{*,q}, p_2^{*,q}, \cdots, p_m^{*,q}\big)$ for $\forall\, q$. Conditions (10), (11) holding for $\forall\, q$ guarantee subroutine $T\big(t, x^q, \lambda^q(p^q)\big)$ converges to the unique open-loop Nash equilibrium $p^{*,q}$, $\forall\, q$.

By the assumption on the order of revenue rate functions for different classes, it is never optimal for a higher class unit to be priced at a lower class fare unless there are enough higher class units to run at its revenue rate, when an estimated overstock of $x_i^q - \lambda_i^{*,q} t$ units of class q, $\forall\, q$ can be put on sale at the price of class $q+1$ for any firm i. Thus, Algorithm 1 solves for the differential game (35)-(36) and the result of (relative) ϵ Nash equilibrium follows from Theorem 5 and 6. $\square$

Algorithm 1 also solves for the open-loop Nash equilibrium in a situation that a higher class has higher priority to be fulfilled and only an estimated overstock can be priced at a lower class fare. For example, business class airline seats are reserved for business travelers and not ready to be pooled with coach class seats and priced at a lower fare. In some sense, this procedure of seats allocation is similar to the standard quantity-based RM practice.

Next we consider a generalization of problem (35)-(36) but hierarchy allocation scheme of Algorithm 1 is not available for this general case.

6.2 Competitive Dynamic Pricing over Networks

Consider a set of M firms, $m = 1, 2, \cdots, M$, who produce J types of products, $j = 1, 2, \cdots, J$. To provide a unit of product j, $j = 1, 2, \cdots, J$ requires a_{ij} units of resource i, $i = 1, 2, \cdots, I$. We define the bill of materials matrix $A = (a_{ij}) \in \mathbb{Z}_+^{I\times J}$ and assume A has no zero columns. Each firm m, $\forall\, m = 1, 2, \cdots, M$, has a vector of resource supplies $n_m = (n_m^1, n_m^2, \cdots, n_m^I)^T \in \mathbb{Z}_+^I$. There is a deadline $t > 0$ for every firm after which selling must stop, and no additional resources can be obtained over $[0, t]$. Demand for products at time $s \in [0, t]$ is a multivariate, stochastic point process with Markovian intensities. At any time $s \in [0, t]$, the matrix of demand intensities $\Lambda = (\lambda_m^j) = (\lambda_1, \lambda_2, \cdots, \lambda_M)$, where $\lambda_m = (\lambda_m^1, \lambda_m^2, \cdots, \lambda_m^J)^T$ and λ_m^j is the demand intensity for product j of firm m. We assume at any time $s \in [0, t]$, λ_m^j for $\forall\, m$ and $\forall\, j$, is determined by s and the current price matrix $P = (p_m^j) = (p_1, p_2, \cdots, p_M)$, where $(P)_m := p_m = (p_m^1, p_m^2, \cdots, p_m^J)^T$ and p_m^j is the current price for product j of firm m. We assume the demand intensity functions $\lambda_m(P, s) : \mathbb{R}_+^{MJ+1} \to \mathbb{R}_+^J :$ $(P, s) \mapsto \big(\lambda_m^1(P, s), \lambda_m^2(P, s), \cdots, \lambda_m^J(P, s)\big)$ satisfies the following regularity conditions:

1. For every $s \in [0, t]$, the demand intensity function $\Lambda(P, s)$ has an inverse, denoted by $P(\Lambda, s)$, $P(\cdot, s) : \mathbb{R}_+^{MJ} \to \mathbb{R}_+^{MJ}$.

2. Let $p_{-m} = (p_1, \cdots, p_{m-1}, p_{m+1}, \cdots, p_M)$, $\forall\, m$, denote the pricing matrix of the $m-1$ firms other than firm m. For any given p_{-m}, $\forall\, m$, the inverse of $\lambda_m(p_m; p_{-m}, s)$ is denoted by $p_m(\lambda_m; p_{-m}, s)$. The revenue rate at time s with fixed p_{-m}, defined by $r_m(\lambda_m; p_{-m}, s) := {\lambda_m}^T p_m(\lambda_m; p_{-m}, s)$, is continuous, bounded and concave in λ_m. Further, we assume

$$\lim_{\lambda_m^j \to 0} \lambda_m^j p_m^j(\lambda_m; p_{-m}, s) = 0 \quad \text{for all finite } \lambda_m^j\,, \forall\, j\,, \forall\, m\,,$$

and the maximizer vector $\lambda_m^*(p_{-m}, s)$ of the revenue rate is uniformly bounded for all $s \in [0, t]$ and

p_{-m}.

3. For each product j of any firm m at any time s, there exists a *null* price $p_m^{\infty,j}(p_{-m}, s)$ such that if $\{p_k\}$ is any sequence of price vectors satisfying $p_k^j \to p_m^{\infty,j}(p_{-m}, s)$, then $\lim_{k\to\infty} \lambda_m^j(p_k; p_{-m}, s) = 0$.

We formulate the nonzero-sum differential game as follows: there is an initial resource matrix $X = (x_m^i) = (x_1, x_2, \cdots, x_M) \in \mathbb{Z}_+^{I\times M}$ and a finite time $t > 0$ to sell the line of products $j = 1, 2, \cdots, J$ in continuous amounts. We denote $\mathcal{V}$ the joint allowable pricing policy space. Given a pricing policy $P = (p_m, p_{-m}) \in \mathcal{V}$, an initial resource matrix X and a finite horizon t, we denote the expected profit for firm m, $\forall\, m = 1, 2, \cdots, M$, by

$$J_m(t, x_m, P) = \int_0^t p_m(s)^T \lambda_m(P, s)\, ds\,,$$

and the restrictions on the resource availability are

$$\int_0^t A\lambda_m(P, s)\, ds \le x_m\,, \quad \forall\, m\,.$$

The open-loop Nash equilibrium of this nonzero-sum differential game still can be solved by using the *maximum principle* in optimal control theory but the solution is not necessarily to be a fixed pricing policy since the demand intensity functions are assumed to be time-dependent in this extension. We assume the open-loop Nash equilibrium of pricing strategies is unique and denoted by $\Phi(s) = \big(\phi_m^j(s)\big) = \big(\phi_1(s), \phi_2(s), \cdots, \phi_M(s)\big) \in \mathcal{V}$ for $s \in [0, t]$, where $\phi_m(s) = \big(\phi_m^1(s), \phi_m^2(s), \cdots, \phi_m^J(s)\big)^T$ for $\forall\, m = 1, 2, \cdots, M$. We denote the corresponding demand intensity matrix by $\Gamma(s) = \big(\gamma_m^j(s)\big)$.

Similar to Gallego and van Ryzin (1997), we suggest the following two heuristics according to the open-loop Nash equilibrium:

Make-to-Stock (MTS) Policy: Firm m, $m = 1, 2, \cdots, M$, preassembles $\lfloor \int_0^t \gamma_m^j(s)\, ds \rfloor$ units of product $j = 1, 2, \cdots, J$ and place the products in separate inventories. Price products at $\phi_m^j(s)$, $s \in [0, t]$ for $\forall\, j$ and sell them until the product inventories are exhausted or the deadline t is reached, whichever comes first.

Make-to-Order (MTO) Policy: Firm m, $m = 1, 2, \cdots, M$, follows the deterministic price path

$\phi_m^j(s)$ over $[0,t]$ for any product j, assembles, sells products in the order in which they are requested and rejects requests for a product j when the inventory of one or more of its resources i drops below a_{ij}.

Theorem 9. *For the nonzero-sum stochastic game over networks, both MTS policy and MTO policy sustain a (relative) ϵ Nash equilibrium in the limiting regime of a sequence of scaled stochastic games with the initial resource matrix $X_k = kX$ and the demand intensity function $\Lambda_k(P,s) = k\Lambda(P,s)$, $k = 1, 2, \cdots$.*

Proof. By Gallego and van Ryzin (1997, Theorem 2,3) and remarks therein, the result follows immediately from the definition of (relative) ϵ Nash equilibrium. □

6.3 Other Extensions

As demonstrated above, the arguments leading to the asymptotic optimality of the open-loop policy can be readily adapted to show that the open-loop Nash equilibrium in the differential game sustains a (relative) ϵ Nash equilibrium for the stochastic game asymptotically. As a result, any dynamic pricing problem discussed in Gallego and van Ryzin (1994, §5), such as compound Poisson process, resupply, cancelations and overbooking, can in principle be extended to a competitive setting as we have done in this paper.

Additionally, for the basic model with holding costs (not necessarily linear) and discounted cash flows, the open-loop Nash equilibrium in the nonzero-sum differential game would be time-dependent, but will still sustain a (relative) ϵ Nash equilibrium asymptotically.

As seen in §6.1, segmentation leads to a fixed price for each fare class in equilibrium over periods where demand intensities are time independent. This result extends to sales channels with a fixed pricing policy for each sale channel and fare class combination over intervals where demand intensities are time independent. The best response for firms with limited capacities is a marginal analysis allocating resource to different channels and fares. The best response problem for firms with abundant inventories is to use revenue-rate maximizing prices for each channel and fare class. If the existence and uniqueness sufficient conditions hold for the demand intensity function of each channel and fare, we can use the tatônnement best-response scheme to compute the open-loop Nash equilibrium for the problem with multi-channels. In practice, we can model and estimate price sensitivities by channel with customers searching through the Internet perhaps being more price sensitive (exhibiting *priceable* demand; Boyd and Kallesen (2004)) than customers searching directly through a carrier's call center, a tour operator or a corporate account (exhibiting *yieldable* demand; Boyd and Kallesen (2004)).

7 Conclusions

We have shown how a range of inventory pricing problems under competition can be formulated as intensity control games and analyzed by considering the corresponding differential game. It is encouraging that the existence and uniqueness of the open-loop Nash equilibrium follows for most of the commonly used demand intensity functions under natural assumptions on the parameters. Not only is the unique equilibrium easy to compute by adaptive learning algorithms, but the use of such heuristics appear to work well in most instances. We strongly believe that this class of competitive inventory pricing models should be topics of intense interest to managers in a wide range of industries.

On the methodological level, this paper relates the stochastic game and its deterministic counterpart. No doubt other variants of the problem can be attacked using the same approach. Indeed, we expect to find Nash equilibria, rather than ϵ-Nash equilibria under additional assumptions to the basic model, e.g., for the markup or the markdown problem with irreversible price changes and a fixed price menu.

In practice, we divide the planning horizon into intervals and assume each firm has a piecewise *regular* demand intensity function. It can be proved that the open-loop Nash equilibrium in the corresponding differential game must be a piecewise fixed pricing policy. Under conditions to guarantee pointwise supermodularity, the tatônnement best-response scheme converges to the open-loop Nash equilibrium. The best-response problem for those firms with limited inventories is a marginal analysis problem while the best-response strategy for those firms with abundant inventories is just to use the maximizer of the revenue rate at any time. We will prove and demonstrate these points numerically in a separate paper using real data from airline industry.

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